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(4,8) Ordered Generalized Whist Designs - Existence Results

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Abstract

Ordered Generalized Whist tournaments are a new design. In this study, we establish that ordered generalized whist tournaments exist for tournaments with $8n + 1$ players, n odd, with 4 players per team.

1 Introduction

In this paper we introduce a new specialization of a design that has been around for many years. We call this specialization an ordered generalized whist tournament.

A standard whist tournament has $4n$ or $4n + 1$ players in all. A generalized whist can include $v = 0, 1 \pmod{k}$ players, where k is the number of players per game. In all tournaments, any pair of players must partner and oppose one another a fixed number of times depending on the number of players.

Whist tournaments are designed around a table of players. The exact number of players(v) and how many tables of players, called a "game", per round depends on the initial parameters. For every round, every player, with the exception of the player represented by that round number, participates in the round.

In the standard whist tournament for $4n$ or $4n + 1$ players, any pair of players partner once and oppose twice throughout the course of tournament. One specialization of a whist tournament is an ordered whist tournament. If you think of 4 players sitting around a circular table in such a tournament, the way in which any pair of players oppose each other is balanced, in that they oppose once while sitting in a N or S position and once while sitting in an E or W position.

Example 1.1. *An ordered whist tournament, Owh(5), is given by the following rounds: (1, 2, 4, 3), (2, 3, 0, 4), (3, 4, 1, 0), (4, 0, 2, 1), (0, 1, 3, 2)*

1 opposes 2 while sitting in the north position in round 0 and also opposes 2 in round 3 from the west position. 1 also partners 2 once, in round 4.

Whist tournaments are known to exist for all $4n$ and $4n + 1$ players and ordered whist tournaments have been proven to exist for all $v = 4n$ or $4n + 1$ players [1] [2].

We wanted to balance a (4,8) generalized whist. In order to understand this design, we define a standard generalized whist tournament.

Definition 1.1. *[3] Let e,k,t,v be positive integers such that k=et and v=0,1 (mod k). A (t,k) Generalized Whist Tournament Design on v players is a group of games in which e teams of t players each compete simultaneously.*

(i) A schedule of games for a tournament involving v players to be played in v rounds

(ii) A game involves k players, with teams of t players competing

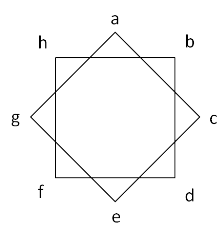
(iii) A round consists of (v-1)/k simultaneous games, with a player playing in all but 1 round

(iv) Every player partners every other player t-1 times

(v) Every player opposes every other k-t times

The teammate and opponent balance requirements are referred to as the Generalized Whist Conditions.

The notation (t,k)GWhD(v) means (t players per team, k players per game) GWhD(total number of players). Thus, (4,8)GWhD(v) are tournaments with games of 8 players, with 2 teams of 4, (partnering 3 times and opposing 4 times). We denote a game of the tournament (a, b, c, d, e, f, g, h) where the team consisting of a, c, e, g opposes the team consisting of b, d, f, h.



1.1 Tournament Design

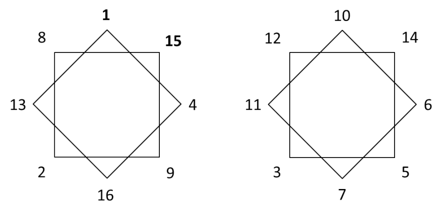
In this study, we focus on Z-cyclic tournaments, which means the players can be thought of as elements of Z_n and every round can be obtained by adding 1 $(mod\ v)$ to the previous round, or by adding the desired round number to round 0.

An important aspect of this design is based on differences. There are two kinds of differences: partner and opponent. Partner differences are obtained by taking a player and subtracting their partners. This is done with all 8 players at the initial round table. So, 8 players, each with 3 partners, results in 24 partner differences for each game.

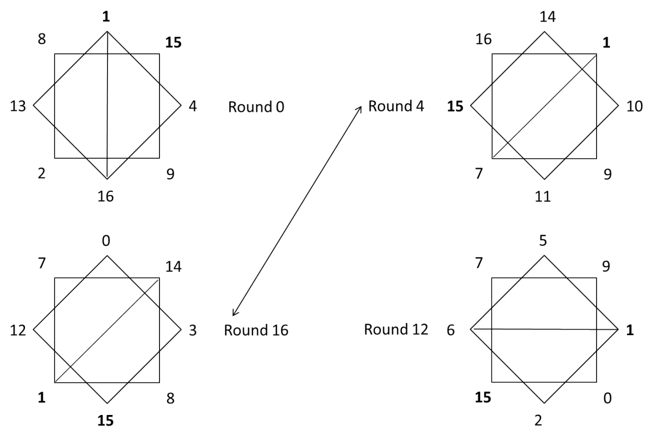
Opponent differences are calculated by taking the players along that diagonal and subtracting their opponents from both of them. This is done for every game in the round. As can be seen from the Theorem in the next section, if all non-zero elements are present in the set of differences, then every player is opposed that round from that position.

Let us begin with an example of a traditional (4,8)GWhD(v).

Example 1.2. *(4,8)GWhD(17)*
Rounds consist of 2 games, of which Round 0 consists of (1, 15, 4, 9, 16, 2, 13, 8); (10, 14, 6, 5, 7, 3, 11, 12) as pictured below. The focus will be on players 1 and 15.



After generating all the rounds in the tournament, we see that 1 and 15 are also opponents in rounds 4, 12, and 16. Here we examine those particular games.

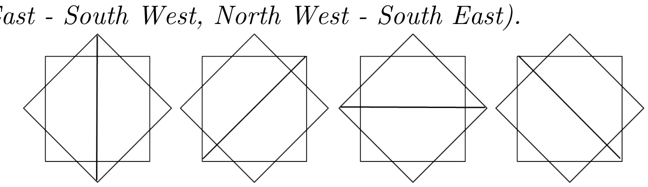


Notice 1 opposes 15 from the same diagonal twice, we wanted to determine if we could "balance" the 4 times any pair of players meet as opponents.

2 Main Results

Motivated by a desire to balance a generalized whist tournament, we define the new concept of an ordered generalized whist tournament as follows:

Definition 2.1. *An ordered generalized whist tournament for 8n+1 players with teams of size 4 and games of size 8 is: a generalized whist tournament (defined above) in which the four times any pair of players meet as opponents, they oppose once each from cardinal directions (North-South, East-West, North East - South West, North West - South East).*



Ordered Generalized Whist tournament Designs have not been looked at before.

The goals for this project are:

(i) To find an ordered (4,8) Generalized Whist Tournament

(ii) To create a theorem to determine conditions on which a (4,8) generalized whist tournament is ordered

(iii) To create a construction for such a tournament

Theorem 2.1. *Let G be an abelian group such that $|G| = 8n+1$. Let $(a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i)$ denote non-identity elements in G. Suppose that the collection of games $(a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i)$, $0 \le i \le n-1$, constitutes an initial round of a cyclic GWhD(8n+1) (i.e. for every $g \in G$, the round in which g sits out is obtained by adding g to every element in the initial round). This GWhD(v) is ordered if and only if the following holds*

$$\bigcup_{i=0}^{n-1}\{(a_i-b_i),(a_i-d_i),(a_i-f_i),(a_i-h_i),(e_i-b_i),(e_i-d_i),(e_i-f_i),(e_i-h_i)\} = G-\{e\},$$

$$\bigcup_{i=0}^{n-1}\{(c_i-b_i),(c_i-d_i),(c_i-f_i),(c_i-h_i),(g_i-b_i),(g_i-d_i),(g_i-f_i),(g_i-h_i)\} = G-\{e\},$$

$$\bigcup_{i=0}^{n-1}\{(b_i-a_i),(b_i-c_i),(b_i-e_i),(b_i-g_i),(f_i-a_i),(f_i-c_i),(f_i-e_i),(f_i-g_i)\} = G-\{e\},$$

$$\bigcup_{i=0}^{n-1}\{(d_i-a_i),(d_i-c_i),(d_i-e_i),(d_i-g_i),(h_i-a_i),(h_i-c_i),(h_i-e_i),(h_i-g_i)\} = G-\{e\},$$

where e is the identity of G and the above differences are referred to as the North-South, East-West, North East - South West, North West - South East opponent differences.

Proof. ($\Leftarrow$) Since G has order $8n+1$, G has $8n$ distinct nonidentity elements, therefore, the $8n$ North-South, East-West, North East - South West, North West - South East differences are all distinct.

Assume that the tournament is not ordered. Then there exists at least one pair (x, y) having the property that in their four meetings as opponents, x, say, opposes y while sitting in the same type of position more than once.

Without loss of generality, suppose they meet twice in the NS position. Suppose they meet in round i and round j as $(x, y, *, !, :, ', /, +)$ and $(+, ', /, !, x, y, *, :)$.

All games are translates of games in the initial round $(a_0, b_0, c_0, d_0, e_0, f_0, g_0, h_0)$
$x - y = a_i - b_i$ for some initial round table $(a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i)$
$x - y = e_j - b_j$ for some initial round table $(a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j)$

Therefore $x - y = x - y \to a_i - b_i = e_j - b_j$, which contradicts the fact that the differences are distinct.

($\Rightarrow$) Suppose that the GWhD($8n+1$) is ordered. Without loss of generality, assume that

$$N{-}S \bigcup_{i=0}^{n-1}\{(a_i-b_i),(a_i-d_i),(a_i-f_i),(a_i-h_i),(e_i-b_i),(e_i-d_i),(e_i-f_i),(e_i-h_i)\} \ne G-\{e\}$$

Since no difference can equal the identity, this latter assumption implies that at least two differences have the same value. No two differences from the same initial round table can be equal without violating the properties of a GWhD($8n+1$). Thus if two differences are equal, they must come from distinct initial round tables.
Say table $i = (a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i)$ and table $j = (a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j)$.

Suppose that $a_i - b_i = e_j - b_j$. Define x by the requirement that $e_j + x = a_i$. In round x, table j becomes $(a_j+x, b_j+x, c_j+x, d_j+x, e_j+x, f_j+x, g_j+x, h_j+x)$ which is equivalent to $(a_j + x, b_i, c_j + x, d_j + x, a_i, f_j + x, g_j + x, h_j + x)$.

Comparing table i with table j, we see a_i opposes b_i twice while sitting NS, contradicting the fact that the tournament is ordered. $\square$

2.1 Construction

We focus on the case when the number of players, $v = 8n + 1$ is a power of a prime, since we are guaranteed that each of the players in the tournament which are thought of as elements of $Z_v - \{0\}$ can be expressed as powers of a primitive root.

Since we are concerned with the case where $p = 8n + 1$ is a prime, we can represent the elements of $Z_p - \{0\}$ as powers of a primitive root x. $p = 8n+1 = 2^k t + 1, k \geq 3$.

If an element of $Z_p - \{0\}$ can be expressed as x^i and $i \equiv j \pmod{2^k}$, then we say that x is in the i^{th} cyclotomic class of index 2^k. We can express the $8n + 1$ players in 2^k cyclotomic classes.

Example 2.1. *The following table shows the elements of $Z_{41} - \{0\}$ expressed as powers of 6, a primitive root of 41, and arranged into $2^3 = 8$ cyclotomic classes:*

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*
6^0	6^1	6^2	6^3	6^4	6^5	6^6	6^7
6^8	6^9	6^{10}	6^{11}	6^{12}	6^{13}	6^{14}	6^{15}
6^{16}	6^{17}	6^{18}	6^{19}	6^{20}	6^{21}	6^{22}	6^{23}
6^{24}	6^{25}	6^{26}	6^{27}	6^{28}	6^{29}	6^{30}	6^{31}
6^{32}	6^{33}	6^{34}	6^{35}	6^{36}	6^{37}	6^{38}	6^{39}

Partners are players who are from the 0^{th}, 2^{nd}, 4^{th}, and 6^{th} class or the 1^{st}, 3^{rd}, 5^{th}, and 7^{th} class.

Theorem 2.2. *Let $p = 8n + 1 = 2^k t + 1, k \geq 3$ be a prime, x be a generator of $Z_p - \{0\}$, y be an even power of x, and $m = 2^{k-2}$, $-x(-1 + x^m)$ be in the $j + 2m^{th}$ class, $0 \leq j < 2^k$, $(-1 + x^m)y$ be in the aa^{th} class, $0 \leq aa < 2^k$, and $(1 + x^m + x^{2m}) = l$ be in the m^{th} or $3m^{th}$ class.*

If the classes in the table below are satisfied in addition to the partner conditions, then

$$(x, x^{m+1}, x^{2m+1}, x^{3m+1}, y, x^m y, x^{2m} y, x^{3m} y) \times x^{8i}, 0 \leq i \leq n-1,$$

is the initial round of a cyclic (4,8)OGWhD($8n + 1$).

	ac-bd		*ab-cd*	
	$l = m$	$l = 3m$	$l = m$	$l = 3m$
$x^{m+1} - y$	$aa + 2m$		$j + 2m$	
$-x + x^m y$	$j + 2m$		$aa + 2m$	
$-x^{3m+1} + y$	$aa + m/j + m$	$aa + 3m/j + 3m$	$j + m/aa + m$	$aa + 3m/j + 3m$
$x - x^{3m} y$	$j + m/aa + m$	$j + 3m/aa + 3m$	$aa + m/j + m$	$j + 3m/aa + 3m$

It is important to note that "ac-bd" and "ab-cd" represent 2 ways of pairing players based on their cyclotomic classes. These pairings are shown below.

Proof. Using the construction and Theorem 2.1, the opponent differences are shown below under the conditions of Theorem 2.2. We see we satisfy the conditions of Theorem 2.1.

	North-South	NorthEast-SouthWest	East-West	NorthWest-SouthEast
a	$-x(-1+x^m)$ $-x(-1+x^m)(1+x^m+x^{2m})$	$x(-1+x^m)$ $-x^{m+1}(-1+x^m)$	$x^{m+1}(-1+x^m)$ $-x^{2m+1}(-1+x^m)$	$x(-1+x^m)(1+x^m+x^{2m})$ $x^{2m+1}(-1+x^m)$
b	$x - x^m y$ $-x^{3m+1} + y$	$x^{m+1} - y$ $-x^m(x^{m+1} - y)$	$x^m(x^{m+1} - y)$ $-x^{2m}(x^{m+1} - y)$	$x^{3m+1} - y$ $x^{2m}(-x + x^m y)$
c	$x - x^{3m} y$ $-x^{m+1} + y$	$-x^m(-x + x^m y)$ $-x + x^m y$	$-x^{2m}(-x + x^m y)$ $x^m(-x + x^m y)$	$x^{2m}(x^{m+1} - y)$ $-x + x^{3m} y$
d	$-(-1+x^m)y$ $-(-1+x^m)(1+x^m+x^{2m})y$	$(-1+x^m)y$ $-x^m(-1+x^m)y$	$x^m(-1+x^m)y$ $-x^{2m}(-1+x^m)y$	$(-1+x^m)(1+x^m+x^{2m})y$ $x^{2m}(-1+x^m)y$

This ensures that there is only one in each of the 2^k cyclotomic classes.
It is also possible to swap one or both of the following:

- $-x^{3m+1} + y$ and $x - x^{3m} y$
- $(-1+x^m)(1+x^m+x^{2m})y$ and $x(-1+x^m)(1+x^m+x^{2m})$

□

Example 2.2. *This is the first known example of a (4,8) ordered generalized whist tournament.*

The initial round consists of $(250, 40, 69, 299, 312, 100, 16, 278) \times x^{8j}, 0 \leq j \leq n-1$*, found using the construction with* $p = 313$*,* $x = 250$*, and* $y = 312$*. It can be verified with the theorem in the previous section. The players are in the following classes:*

0	1	2	3	4	5	6	7
1	250	213	40	297	69	35	299
256	148	66	224	286	136	196	172
119	15	307	65	287	73	96	212
103	84	29	51	230	221	162	123
76	220	225	223	36	236	156	188
50	293	8	122	139	7	185	239
280	201	170	245	215	227	97	149
3	124	13	120	265	207	105	271
142	131	198	46	232	95	275	203
44	45	295	195	235	219	288	10
309	252	87	153	64	37	173	56
228	34	49	43	108	82	155	251
150	253	24	53	104	21	242	91
214	290	197	109	19	55	291	134
9	59	39	47	169	308	2	187
113	80	281	138	70	285	199	296
132	135	259	272	79	31	238	30
301	130	261	146	192	111	206	168
58	102	147	129	11	246	152	127
137	133	72	159	312	63	100	273
16	244	278	14	57	165	247	89
27	177	117	141	194	298	6	248
26	240	217	101	210	229	284	262
83	92	151	190	237	93	88	90
277	77	157	125	263	20	305	191
174	306	128	74	33	112	143	68
98	86	216	164	310	189	300	193
48	106	208	42	171	182	115	267
81	218	38	110	269	268	18	118
78	94	25	303	4	61	226	160
249	276	140	257	85	279	264	270
205	231	158	62	163	60	289	260
209	292	71	222	99	23	116	204
294	258	22	179	304	254	274	266
144	5	311	126	200	233	32	175
243	28	114	17	181	178	54	41
234	282	75	283	12	183	52	167
121	202	107	145	255	211	166	184
302	67	161	186	176	180	241	154

3 Conclusion

The concept of rigid motion is highly beneficial when considering how many unique ways of reordering the construction exist. As games consist of 8 players, in 2 teams of 4, we examine squares. A single square allows for 4! = 24 ways to order 4 players. This means there could potentially be 576 possible ways to order both teams of players. The octic group, the group based on the transformations of a square, contains 8 elements. Transformations don't change the diagonals so the partners remain the same. Thus, 24/8 = 3, implying that there are only 3 unique partner positions for a single table. Involving two tables, this means a total of only 3*3 = 9 unique possible starting positions that keep partners the same, but alters their position around the table. In this paper, we only explored 1 of these.

The number of players as well as t and k could be changed and this concept can be easily extended to other designs.

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