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Monty Hall Meets Game Theory

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MONTY HALL MEETS GAME THEORY

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Abstract

In 1990, newspaper columnist Marilyn vos Savant posed the following puzzle to her column:

"Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the other doors, opens another door, say No. 3, which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to take the switch?" [6].

She included the answer for the puzzle, but received much push-back from mathematicians across the country, baffled that she described how switching doors would give you an advantage to win, rather than it being a 50-50 chance since there are only two doors in play. Since then, mathematicians have been studying this phenomena and were actually able to prove Ms. vos Savant's answer. In this paper, I will discuss the contestant's chances of winning and how it is affected by them staying or switching their door, and also how an increase of doors affect their chances of winning as well. I will use probability as well as game theory in order to support these answers. Proving the Classic Monty case is guided by the work of Gabriel Sandu in [5], as well as Stephen Lucas, Jason Rosenhouse, and Andrew Schepler in [2]. I will also briefly discuss a different version of the Monty Hall game and how it changes the efficacy of the stay/switch strategies.

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Contents

1 Introduction: Classic Monty

The Monty Hall problem is based off of a well-known segment of the game show *Let's Make a Deal*, hosted by Monty Hall. During one portion of the show, the contestant is shown 3 doors, two of which conceal a low-value prize (a "zonk") and the remaining door concealing a high-value prize (a car or an expensive piece of furniture, for example). Monty Hall would first hide the prize behind one of the three doors. After he lets the contestant pick a door, he opens a door to reveal a "zonk". He gives the contestant the option to stay with their door or switch to the other door that hasn't been revealed. Once they pick, Monty lets them open their door.

There are some rules and considerations for this game:

1. After the contestant chooses a door, Monty can not open the door with the prize, nor the door the contestant has chosen.

2. Once Monty has opened a door, the contestant is given an option to stay with their initial door choice or switch to the other unopened door. These stay/switch choices will be the strategies of the contestant during this game.

3. For n doors with $n > 3$, when we discuss the switch strategy, we will mean that the contestant will stay with their door until there are two doors left unopened, and by then the contestant will choose to switch their door.

These, in turn, will become their respective strategies. For clarity, in our version of the game, we will designate the "zonk" prize to be a goat and the high-value prize to be a car. So, given that the contestant is aware of Monty's door-opening restrictions, our question becomes: What is the contestant's optimal strategy for winning?

2 The Probability Approach

2.0.1 The 3-door case

Lucas, Rosenhouse, and Schepler in [2] provide a thorough explanation of how the probability changes after a contestant chooses a door and Monty opens a different door to reveal a goat. Without loss of generality, assume the contestant chooses Door 3, and let B represent the event where Monty opens Door 2. Let D_i be the prize being hidden behind Door i. Note that the game assumes equal probability for the car to be hidden behind any of the three doors; that is,

$$P(D_1) = P(D_2) = P(D_3) = \frac{1}{3}.$$

Now, suppose we want to calculate the probability of the car being behind a door, given that Monty has opened one of the three doors. This is called conditional probability, which means finding the probability of any event A to happen, conditioning on the fact that event B has already occurred, and we represent this as $P(A|B)$. This probability is calculated using Bayes' Theorem as follows:

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)},$$

where $P(B) = \sum_{i=1}^{n} P(A) \times P(B|A)$.

For our scenario, the formula looks like the following:

$$P(D_i|B) = \tfrac{P(B|D_i) \times P(D_i)}{P(B)},$$

with $P(D_i|B)$ representing the probability of having the prize behind door D_i knowing that Monty has revealed door 2. Thus, we can use this formula to find the two probabilities $P(D_1|B)$ and $P(D_3|B)$.

We adhere to the following rules in our calculations of each probability:

- $P(B|D_i)$ is the probability of Monty opening Door 2, given that the car is behind door i. Therefore,
  - If the car is behind Door 1, then the probability of Monty opening Door 2 given that the car is behind Door 1, $P(B|D_1)$, is 1. This is because Monty can't choose the door with the car and can't open the door that the contestant chose, leaving him only with one option.
  - If the car is behind Door 2, then $P(B|D_2) = 0$, since Monty cannot open the door with the car.
  - If the car is behind Door 3, then $P(B|D_3)$ is $\frac{1}{2}$ since he can open either door 1 or 2 with equal probabilities.
- $P(D_i) = \frac{1}{3}$ for all i.
- Since $P(B) = \sum_{i=1}^{n} P(D_i) \times P(B|D_i)$ and $n = 3$. Then,

$$\begin{aligned} P(B) &= P(D_1) \times P(B|D_1) + P(D_2) \times P(B|D_2) + P(D_3) \times P(B|D_3) \\ &= \frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2} \\ &= \frac{1}{3} + 0 + \frac{1}{6} = \frac{1}{2} \end{aligned}$$

Next, to calculate the probabilities of having the car behind door D_i knowing that Monty has revealed Door 2, we substitute the above values into Bayes' Theorem and achieve the following results:

$$\begin{aligned} P(D_1|B) &= \frac{P(B|D_1) \times P(D_1)}{P(B)} = \frac{1 \times 1/3}{1/2} = \frac{2}{3}, \\ P(D_2|B) &= \frac{P(B|D_2) \times P(D_2)}{P(B)} = \frac{0 \times 1/3}{1/2} = 0, \\ P(D_3|B) &= \frac{P(B|D_3) \times P(D_3)}{P(B)} = \frac{1/2 \times 1/3}{1/2} = \frac{1}{3}. \end{aligned}$$

Therefore, since $P(D_1|B)$ is larger than the other two probabilities, we conclude that the contestant has a higher probability of winning the car if they switch to Door 1 rather than sticking to their original choice of Door 3. Thus, the switching strategy is the optimal one for the player.

In the next section we study the 4-door case and report on whether we reach the same conclusion or not.

2.0.2 The 4-door case

We continue using same assumptions, such as the contestant initially picking Door 3 while Monty revealing Door 2 first. Similar to the 3-door case, the probability of the car being initially hid behind any of the doors is still equal, except now it looks like:

$$P(D_1) = P(D_2) = P(D_3) = P(D_4) = \frac{1}{4}.$$

Next, We use Bayes' Theorem to find the probability of having the car in the back of Door D_i given that Monty has revealed what's behind Door 2. However, we first lay out the information we need to know to complete those calculations:

- Recall, $P(B|D_i)$ represents the probability of Monty opening Door 2, given that the car is on the other side of Door i. Therefore, we have a few cases to unfold:
  - If the car is behind Door 1, then $P(B|D_1) = \frac{1}{2}$. This is a result of the fact that Monty can't choose the door with the car neither can he reveal the chosen door by the contestant. Thus, he ends up with two choices either disclosing Door 1 or Door 4, which both are equiprobable.

- Since the rules of the game state that Monty cannot open the door with the prize, then $P(B|D_2) = 0$ if the car is behind Door 2.
- The hidden car behind Door 3 gives Monty three choices of doors, namely D_1, D_2, D_4, with equal probabilities. Thus, $P(B|D_3) = \frac{1}{3}$.
- Similar to case 1, when the car is placed on the other end of Door 4, then $P(B|D_4) = \frac{1}{2}$.

- Recall, in this version of the game, $P(D_i)$ will be equal to $\frac{1}{4}$ for $i = 1, 2, 3, 4$.
- To calculate the probability that Monty would reveal door 2, we use the equation $P(B) = P(D_1)P(B|D_1) + P(D_2)P(B|D_2) + P(D_3)P(B|D_3) + P(D_4)P(B|D_4)$. Thus,

$$\begin{aligned} P(B) &= \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times 0 + \frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{2} \\ &= \frac{1}{3}. \end{aligned}$$

As we search to find the optimal strategy for the contestant to win the big prize, we use Bayes' Theorem to calculate the probability of each strategy:

$$\begin{aligned} P(D_1|B) &= \frac{P(B|D_1) \times P(D_1)}{P(B)} = \frac{1/2 \times 1/4}{1/3} = \frac{3}{8}, \\ P(D_2|B) &= \frac{P(B|D_2) \times P(D_2)}{P(B)} = \frac{0 \times 1/4}{1/3} = 0, \\ P(D_3|B) &= \frac{P(B|D_3) \times P(D_3)}{P(B)} = \frac{1/3 \times 1/4}{1/3} = \frac{1}{4}, \\ P(D_4|B) &= \frac{P(B|D_4) \times P(D_4)}{P(B)} = \frac{1/2 \times 1/4}{1/3} = \frac{3}{8}. \end{aligned}$$

As a result, when the contestant follows the stay strategy, i.e, staying with their initial selection they can win the prize one fourth of the times. However, the switch strategy gives the contestant three fourth of the times, (combining the probabilities of 3/8), to win the car. Hence, we again conclude that the switch strategy is the winning one.

2.0.3 The 5-door case

In this section, We also maintain the supposition that initially the contestant selecting Door 3 and then Monty unlocking Door 2. Similar to the previous two cases, we also assume that each door has an equiprobable chance to have the prize placed behind it. Therefore,

$$P(D_1) = P(D_2) = P(D_3) = P(D_4) = P(D_5) = \frac{1}{5}.$$

Before presenting the probability of each strategy via Bayes' theorem, we carry out the following computations:

- Knowing that $P(B|D_i)$ is the probability of Monty displaying what behind Door 2, given that the car is behind Door i, we have the following cases to consider:
  - As Monty can not reveal the door of the prize and the selected door by the contestant, then he has the choice of uncovering either Doors 2, 4, or 5 when the car is hidden behind Door 1. Therefore, $P(B|D_1) = \frac{1}{3}$.
  - Again $P(B|D_2) = 0$ because Monty cannot open the door with the car.
  - If the car is behind Door 3, $P(B|D_3) = \frac{1}{4}$ as he can choose between Doors 1, 2, 4 , or 5 to give out.
  - Having the car on the other side of Door 4 gives Monty the option to disclose either Doors 1, 2, or 5 which means $P(B|D_4) = \frac{1}{3}$.

– Similarly hiding the prize beyond Door 5 leads to $P(B|D_5) = \frac{1}{3}$.

- Additionally, $P(D_i) = \frac{1}{5}$ for all i.
- The probability that Monty would choose Door 2 to showcase is

$$\begin{aligned}P(B) &= P(D_1) \times P(B|D_1) + P(D_2) \times P(B|D_2) + P(D_3) \times P(B|D_3) \\ &+ P(D_4) \times P(B|D_4) + P(D_5) \times P(B|D_5) \\ &= \frac{1}{5} \times \frac{1}{3} + \frac{1}{5} \times 0 + \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{3} + \frac{1}{5} \times \frac{1}{3} \\ &= \frac{1}{4}.\end{aligned}$$

Next, we compute the conditional probabilities of the prize being at the back of Door i knowing that Monty has chosen to unveil Door 2:

$$P(D_1|B) = \frac{P(B|D_1) \times P(D_1)}{P(B)} = \frac{1/3 \times 1/5}{1/4} = \frac{4}{15},$$

$$P(D_2|B) = \frac{P(B|D_2) \times P(D_2)}{P(B)} = \frac{0 \times 1/5}{1/4} = 0,$$

$$P(D_3|B) = \frac{P(B|D_3) \times P(D_3)}{P(B)} = \frac{1/4 \times 1/5}{1/4} = \frac{1}{5},$$

$$P(D_4|B) = \frac{P(B|D_4) \times P(D_4)}{P(B)} = \frac{1/3 \times 1/5}{1/4} = \frac{4}{15}$$

$$P(D_5|B) = \frac{P(B|D_5) \times P(D_5)}{P(B)} = \frac{1/3 \times 1/5}{1/4} = \frac{4}{15}.$$

From the above equations, the probability of winning the car will be $\frac{4}{15} + \frac{4}{15} + \frac{4}{15} = \frac{4}{5}$ when the contestant decide to switch their initial selection. However, they only have a chance of $\frac{1}{5}$ to win when they keep with the same door as they started. This result once again supports the conclusion that the switching strategy is the winning strategy for the contestant.

3 The Game Theory Approach

3.1 A Brief Background on Game Theory

Game Theory is a discipline of mathematics that studies the competition and cooperation between various parties, typically in a game-like (win/lose) setting [Peters]. Peters outlines a general timeline for the development of game theory. These situations appeared in works as early as the Bible, and after that in the work of Chinese philosopher Sun Tzu around 2,000 years ago, where he modelled strategic warfare using game theory. In 1913, Zermelo produced one of the first formal works on game theory, proving that in the game of Chess, either white has a winning strategy, black has a winning strategy, or each player can call a draw. In 1928, von Neuman proved the minimax theorem for zero-sum games, used later in this paper. This theorem was considered the starting point for game theory and laid the foundation for the study of cooperative games. This also led to the implication of game theory to economics, where Nash (Nash equilibrium), Harsanyi (modelling games with incomplete information), and Selten (subgame perfect Nash equilibrium) all received the Nobel prize in economics in 1994.

Every game must consist of these four qualities:

1. Who is playing?
2. What are they playing with?
3. When does each player get to play?

4. How much do they gain/lose by making the choices they make in the game?

In every game, it is also assumed that there is common knowledge about the rules, meaning that if you were to ask any player in the game what the rules are, you would get the same responses. We will be applying these qualities of game theory to our Monty Hall game in order to find the optimal strategy for the contestant.

We now outline some of the game theory definitions and theorems that will help us in solving the Monty Hall problem.

Definition 1 *A mixed strategy in a two-players game is a probability distribution over either the row or column in a bimatrix i.e. each player follows a certain probability distribution.*

Definition 2 *A Nash equilibrium is when each player has an optimal (winning) strategy given the strategy of the other player. Thus, it consists of a pair of strategies.*

Definition 3 *A bimatrix is a game with entries of the form (a_{ij}, b_{ij}) that is the payoff for each player given that a_{ij} and b_{ij} is the payoff for the column player (the contestant) and the row player (Monty Hall) respectively.*

Theorem 1 *(Von Neuman's Minimax Theorem): Every finite, two-person, win-lose game has an equilibrium in mixed strategies.*

The minimax theorem guarantees that the game has a value, which is the payoff that both players would get following their mixed strategies. In other words, Theorem 1 asserts that an equilibrium always exists and that any two mixed strategy equilibria deliver the same expected utility.

Definition 4 *By a finite two-person game we mean the sequence $(S_I, S_{II}, u_I, u_{II})$ with S_I and S_{II} being the sets of strategies for players I and II respectively. Additionally, the real numbers u_I and u_{II} represent the payoff for player I and II respectively.*

In Monty Hall game, $S_I = S_{MH}$ is the set of strategies for Monty while $S_{II} = S_C$ and $u_{II} = u_C$ are the contestant's set of strategies and payoff, respectively. Thus, the expected utilities for the contestant are defined next.

Definition 5 *The expected utility for the contestant, U_C, in using the stay strategy is given by*

$$U_C(\sigma, \nu) = U_C((y, h_y), \nu) = \sum_{t \in S_{MH}} \nu(t) u_C((y, h_y), t),$$

where σ represents the stay strategy $(y, h_y) = (\textit{initial choice}, \textit{staying with it})$, ν is the probability that runs over all Monty Hall's strategies. Furthermore, $u_c((y, f_y), t)$ represents the payoff (0 or 1) function for the contestant that takes into consideration their initial door choice and their stay strategy as well as where the prize has been hid.

Similarly, the contestant's expected utility, U_C, in using the switch strategy is given by the equation:

$$U_C((y, f_y), \nu) = \sum_{t \in S_{MH}} \nu(t) u_C((y, f_y), t), \quad (1)$$

where (y, f_y) consists of the initial door choice of the contestant and their strategy of switching while $u_c((y, f_y), t)$ represents the payoff function for the strategy.

In order to compute the expected utilities for the contestant, we need to know that the game has an equilibrium. Proposition 1 states the conditions when a game has an equilibrium. Meanwhile, Propositions 2 and 3 explain how to reduce the matrix of a game given there is an equilibrium.

Proposition 1 *In a finite, two player strategic game, the pair (μ^*, ν^*) is an equilibrium if and only if the following conditions hold:*

1. $U_I(\mu^*, \nu^*) = U_I(\sigma, \nu^*)$ *for every* $\sigma \in S_I$ *in the support of* μ^*.

2. $U_{II}(\mu^*, \nu^*) = U_{II}(\mu^*, \tau)$ *for every* $\tau \in S_{II}$ *in the support of* ν^*.

3. $U_I(\mu^*, \nu^*) \geq U_I(\sigma, \nu^*)$ *for every* $\sigma \in S_I$ *outside the support of* μ^*.

4. $U_{II}(\mu^*, \nu^*) \geq U_{II}(\mu^*, \tau)$ *for every* $\tau \in S_{II}$ *outside the support of* ν^*.

Given that $\Gamma = (S_I, S_{II}, \mu_I, \mu_{II})$ is a finite two player, win-lose strategic game, we have:

Definition 6 *We say a strategy* σ' *weakly dominates another strategy* σ *in* S_I *when the utility* $u_I(\sigma', \tau)$ *is greater than or equal to* $u_I(\sigma, \tau)$ *for all* $\tau \in S_I$ *and* $u_I(\sigma', \beta) > u_I(\sigma, \beta)$ *for some* $\beta \in S_{II}$.

Proposition 2 Γ *has an equilibrium in mixed strategies* (μ_I, μ_{II}) *such that for each player p none of the strategies in the support of* σ_p *is weakly dominated in* Γ.

Proposition 3 Γ *has an equilibrium in mixed strategies* (μ_I, μ_{II}) *such that for each player p there are no strategies in the support of* σ_p *which are payoff equivalent.*

In the next three sections we will compute the probability of the winning strategy for the contestant using a game theoretic approach. We start with the 3-door case while keeping the same assumptions as before, i.e. Monty shows Door 2 after the contestant has selected Door 3.

3.1.1 The 3-door case

We first begin by discussing the 3-door case results presented by Gabriel Sandu [3]. Following the same notation as [3], (x, y, z, t) is the sequence of the game with x representing Monty Hall's (MH) door choice to hide the prize, y being the contestant's (C) initial door choice, z is standing for the door MH will open and concealing the "zonk" prize i.e. a goat, and t describing the final door choice of the contestant. For example, the sequence $(3, 1, 2, 3)$ means that MH hid the car behind Door 3, C initially chose Door 1, then MH opened Door 2, and lastly C switched to Door 3. Since $x = t = 3$ in the sequence, then the contestant would win this game and in general the strategy is winning for C when $x = t$ whereas it will be winning for Monty otherwise.

It is worth noting that x and z are moves designated by MH while y and t are moves carried out by C. Additionally, MH plays his move z after the contestant has chosen his initial selection. Therefore, z can be defined as the outcome of a function g_x that relies on both x and y. For example, when $x = 1$, then $g_x = g_1$ and for $y = 2$, the only choice for Monty is to open door 3 i.e. $g_1(1, 2) = 3 = z$.

We summarize the strategy for Monty Hall in Table 1. The first column describes where the car has been hid and all the possibilities g_x. The other columns show the outcomes of g_x after the contestant has selected a door while keeping in mind the door with the prize. For example, $g_2(2, 1) = 3$ represents the event where Monty hides the car behind Door 2 and the contestant chooses Door 1, which forces Monty to open Door 3.

Due to the fact that Monty has two options to choose from when the contestant initially selects the door with the car, we use g_x and g'_x to distinguish between the two cases. We furthermore note that g_x and g'_x are equal when the contestant names a door that doesn't hide the car. In particular, $g_1(1, 2) = g'_1(1, 2) = 3$.

$(1, g_1)$	$g_1(1,1) = 2$	$g_1(1,2) = 3$	$g_1(1,3) = 2$
$(1, g'_1)$	$g'_1(1,1) = 3$	$g'_1(1,2) = 3$	$g'_1(1,3) = 2$
$(2, g_2)$	$g_2(2,1) = 3$	$g_2(2,2) = 1$	$g_2(2,3) = 1$
$(2, g'_2)$	$g'_2(2,1) = 3$	$g'_2(2,2) = 3$	$g'_2(2,3) = 1$
$(3, g_3)$	$g_3(3,1) = 2$	$g_3(3,2) = 1$	$g_3(3,3) = 1$
$(3, g'_3)$	$g'_3(3,1) = 2$	$g'_3(3,2) = 1$	$g'_3(3,3) = 2$

Table 1: MH Strategy

Similar to the probability approach, the contestant C has two strategies: staying (S_C^{Stay}) which means they will stick to their initial selection until the end of the game. However, the switching (S_C^{Switch}) strategy describes the case when the contestant switches their initial door when there are only two closed doors left in the game. Therefore, we can capture these strategies using a function notation.

For $y = 1, 2, 3$, we define the stay strategy function, h_y, such that $h_y(y, z) = t$ that is the final pick of the door for the contestant after Monty has revealed the goat concealing door. The contestant will win when $h_y(y, z) = t = y = x$ and lose when $h_y(y, z) = t = y \neq x$.

Likewise, let f_y be the switch strategy for the contestant with $f_y(y, z) = t$. Since the contestant wins if they initially guessed wrong then $f_y(y, z) = t = x \neq y$. However, they would lose when $x = y$.

In Table 2, we compute the payoffs (i.e who is winning) for each sequence of the game given that the first row outlines all possible strategies for Monty while the first column sums up all the strategies for the contestant. For example, the pair $(1, 0)$ represents a win for the contestant and $(0, 1)$ represents a win for Monty.

	$(1, g_1)$	$(1, g_1')$	$(2, g_2)$	$(2, g_2')$	$(3, g_3)$	$(3, g_3')$
$(1, h_1)$	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)
$(2, h_2)$	(0,1)	(0,1)	(1,0)	(1,0)	(0,1)	(0,1)
$(3, h_3)$	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)
$(1, f_1)$	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)
$(2, f_2)$	(1,0)	(1,0)	(0,1)	(0,1)	(1,0)	(1,0)
$(3, f_3)$	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)

Table 2: Contestant vs MH Strategies

For clarification, the cell at the intersection of $(2, g_2)$ and $(3, h_3)$ describes the payoff of the event where the car is behind Door 2, the contestant chooses Door 3, then Monty reveals Door 1, and the contestant stays at Door 3 (the h-function has the same index as y). As a result of this sequence, Monty wins because the contestant didn't choose the door with the car. Hence, the payoff pair is $(0, 1)$.

From Table 2, the staying strategy, in each row, resulted in 2 out of 6 wining payoff pairs for the contestant, i.e. a probability of $\frac{1}{3}$, while each possibility of the switching strategy has a winning probability of $\frac{2}{3}$ for the contestant.

These results confirm that the switch strategy for the contestant has a higher probability for the payoff as we discovered using the probability approach.

We now use Propositions 2 and 3 to reduce the Table 2 in order to calculate the expected utility of each strategy for the contestant C.

First, we observe that the switch strategy weakly dominates the stay strategy for the contestant. That is, the contestant has more payoff pairs in their favor when comparing (j, f_j) with any stay strategy (i, h_i). For example, the strategy $(1, h_1)$ indicates the contestant choosing and staying with Door 1 which leads to two payoff pairs in their favor $(1, 0)$. Meanwhile, the switch strategy $(2, f_2)$ which represents the event where the contestant chooses Door 2 and switches after Monty opens a door has four payoff pairs in the contestant favor. Therefore, using Proposition 2 shrinks Table 2 by eliminating the rows that correspond to any stay strategy (i, h_i).

	$(1, g_1)$	$(1, g_1')$	$(2, g_2)$	$(2, g_2')$	$(3, g_3)$	$(3, g_3')$
$(1, f_1)$	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)
$(2, f_2)$	(1,0)	(1,0)	(0,1)	(0,1)	(1,0)	(1,0)
$(3, f_3)$	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)

Table 3: Payoff of the weakly dominant strategies

It is worth noting that the term "weakly dominates" does not guarantee that switching will win every time. For instance, if the contestant initially chose the door with the car, then they would lose if they switched doors. To summarize, the switching strategy weakly dominates the stay strategy because switching doors increases, but doesn't guarantee, our chances of winning the prize.

Moreover, we remark that strategies (x, g_x) and (x, g'_x) are payoff equivalent (Table 2 or 3) because no matter what door Monty reveals, the contestant is going to play the same strategy, which leads to the same payoff pair. Thus, using Proposition 3, we can eliminate the (x, g'_x) strategies without changing the game.

	$(1, g_1)$	$(2, g_2)$	$(3, g_3)$
$(1,f_1)$	(0,1)	(1,0)	(1,0)
$(2,f_2)$	(1,0)	(0,1)	(1,0)
$(3,f_3)$	(1,0)	(1,0)	(0,1)

Table 4: Payoff of the weakly dominant strategies without the equivalent MH payoff pair

The author in [5] outlines that when μ and ν are both uniform probability distributions, such that $\mu(y, f_y) = \frac{1}{3}$ and $\nu(x, g_x) = \frac{1}{3}$, then it can be proven using Proposition 1 that the pair of strategic games (μ, ν) is an equilibrium. Therefore, we can calculate the expected utilities using Definition 5 and equation 1.

As a reminder, we are still assuming that the car is hidden behind Door 1, the contestant initially chooses Door 3 and Monty opens Door 2.

Using Table 4 and equation 1, we compute the expected utility for the contestant when they follow the strategy $(1, f_1)$.

$$\begin{aligned}
U_C((1, f_1), \nu) &= \sum_{t \in S_{MH}} \nu(t) u_C((1, f_1), t) \\
&= \nu(1, g_1) u_C((1, f_1), (1, g_1)) + \nu(2, g_2) u_C((1, f_1), (2, g_2)) \\
&+ \nu(3, g_3) u_C((1, f_1), (3, g_3)) \\
&= \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times 1 = \frac{2}{3}
\end{aligned}$$

Similarly, $U_C((y, f_y), \nu) = \frac{2}{3}$ which is the expected utility for the contestant in following any of the switch strategies.

Since Table 4 does not include the stay strategy, we need to calculate the expected utility for this strategy using Table 2. We still consider ν as a uniform distribution with $\nu(x, g_x) = \nu(x, g'_x) = \frac{1}{6}$ while μ is a probability distribution with $\mu(y, f_y) = \frac{1}{3}$ and $\mu(y, h_y) = 0$. Therefore,

$$\begin{aligned}
U_C((1, h_1), \nu) &= \sum_{t \in S_{MH}} \nu(t) u_C((1, h_1), t) \\
&= \nu(1, g_1) u_C((1, h_1), (1, g_1)) + \nu(1, g'_1) u_C((1, h_1), (1, g'_1)) \\
&+ \nu(2, g_2) u_C((1, h_1), (2, g_2)) + \nu(2, g'_2) u_C((1, h_1), (2, g'_2)) \\
&+ \nu(3, g_3) u_C((1, h_1), (3, g_3)) + \nu(3, g'_3) u_C((1, h_1), (3, g'_3)) \\
&= \frac{1}{6} \times 1 + \frac{1}{6} \times 1 + \frac{1}{6} \times 0 + \frac{1}{6} \times 0 + \frac{1}{6} \times 0 + \frac{1}{6} \times 0 = \frac{1}{3}
\end{aligned}$$

As a result, we conclude that the switch strategy has an expected utility that is greater than the stay strategy.

Next, we turn our attention to the 4 and 5-door cases following similar path as in [5].

3.1.2 The 4-door case

We first summarize Monty Hall's strategies in Table 5 keeping the same rules and conditions, except now the S_C^{Switch} strategy means that C is switching their door only when there are two doors left.

$(1,g_1)$	$g_1(1,1)=2$	$g_1(1,2)=3$ $g_1(1,2)=4$	$g_1(1,3)=2$ $g_1(1,3)=4$	$g_1(1,4)=2$ $g_1(1,4)=3$
$(1,g_1')$	$g_1'(1,1)=3$	$g_1'(1,2)=3$ $g_1'(1,2)=4$	$g_1'(1,3)=2$ $g_1'(1,3)=4$	$g_1'(1,4)=2$ $g_1'(1,4)=3$
$(1,g_1'')$	$g_1''(1,1)=4$	$g_1''(1,2)=3$ $g_1''(1,2)=4$	$g_1''(1,3)=2$ $g_1''(1,3)=4$	$g_1''(1,4)=2$ $g_1''(1,4)=3$
$(2,g_2)$	$g_2(2,2)=1$	$g_2(2,1)=3$ $g_2(2,1)=4$	$g_2(2,3)=1$ $g_2(2,3)=4$	$g_2(2,4)=1$ $g_2(2,4)=3$
$(2,g_2')$	$g_2'(2,2)=3$	$g_2'(2,1)=3$ $g_2'(2,1)=4$	$g_2'(2,3)=1$ $g_2'(2,3)=4$	$g_2'(2,4)=1$ $g_2'(2,4)=3$
$(2,g_2'')$	$g_2''(2,2)=4$	$g_2''(2,1)=3$ $g_2''(2,1)=4$	$g_2''(2,3)=1$ $g_2''(2,3)=4$	$g_2''(2,4)=1$ $g_2''(2,4)=3$
$(3,g_3)$	$g_3(3,3)=1$	$g_3(3,1)=2$ $g_3(3,1)=4$	$g_3(3,2)=1$ $g_3(3,2)=4$	$g_3(3,4)=1$ $g_3(3,4)=2$
$(3,g_3')$	$g_3'(3,3)=2$	$g_3'(3,1)=2$ $g_3'(3,1)=4$	$g_3'(3,2)=1$ $g_3'(3,2)=4$	$g_3'(3,4)=1$ $g_3'(3,4)=2$
$(3,g_3'')$	$g_3''(3,3)=4$	$g_3''(3,1)=2$ $g_3''(3,1)=4$	$g_3''(3,2)=1$ $g_3''(3,2)=4$	$g_3''(3,4)=1$ $g_3''(3,4)=2$
$(4,g_4)$	$g_4(4,4)=1$	$g_4(4,1)=2$ $g_4(4,1)=3$	$g_4(4,2)=1$ $g_4(4,2)=3$	$g_4(4,3)=1$ $g_4(4,3)=2$
$(4,g_4')$	$g_4'(4,4)=2$	$g_4'(4,1)=2$ $g_4'(4,1)=3$	$g_4'(4,2)=1$ $g_4'(4,2)=3$	$g_4'(4,3)=1$ $g_4'(4,3)=2$
$(4,g_4'')$	$g_4''(4,4)=3$	$g_4''(4,1)=2$ $g_4''(4,1)=3$	$g_4''(4,2)=1$ $g_4''(4,2)=3$	$g_4''(4,3)=1$ $g_4''(4,3)=2$

Table 5: MH strategy in 4-door game

In this game, since there are 4 doors, Monty Hall has more choices of doors to reveal after the contestant has picked their door; more specifically, Monty has at least two doors to choose from.

As we did in the 3-door case, we use the single and double primes in order to differentiate between events where the contestant chooses the door that is hiding the car. This scenario gives Monty three different doors to showcase in the second round of the game. We also note that $g_x = g'_x = g''_x$ when the contestant select a non-prize door.

To help reading Table 5, let's consider the rows where the car has be placed behind Door 2, namely, the rows $(2, g_2)$, $(2, g'_2)$ and $(2, g''_2)$. The function $g_2(2, 2) = 1$ means that Monty decided to disclose Door 1 when the contestant chose the prize holding Door 2. Likewise, $g'_2(2, 4) = 3$ represents the event where Monty hid the car behind Door 2, the contestant chose Door 4, and Monty would open Door 3.

The matrix representation of the game is given in Table 6. Likewise the 3-door case, h_y and f_y represent the stay and switch strategy respectively. As a reminder, since the number of doors $n > 3$, "switching" means the contestant stays with their door until there are only two doors to choose from at the last round of the game, and then the contestant will switch.

	$(1,g_1)$	$(1, g'_1)$	$(1,g''_1)$	$(2,g_2)$	$(2,g'_2)$	$(2,g''_2)$	$(3,g_3)$	$(3,g'_3)$	$(3,g''_3)$	$(4,g_4)$	$(4,g'_4)$	$(4,g''_4)$
$(1,h_1)$	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)
$(2,h_2)$	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)
$(3,h_3)$	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)
$(4,h_4)$	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)
$(1,f_1)$	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
$(2,f_2)$	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
$(3,f_3)$	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)
$(4,f_4)$	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)

Table 6: Game matrix in 4-door case

We can observe how the stay/switch strategies affect the contestant's chances of winning the car from Table 6. Looking at each h_y row, the contestant only wins $\frac{3}{12}$ (or $\frac{1}{4}$) of the time, however if we look at each row in the f_y case, the contestant wins $\frac{9}{12} = \frac{3}{4}$ of the time. Again, this matches the calculations in Section 2, proving that the contestant has a higher chance of winning if they switch doors when there are 2 doors left. Therefore, (y, f_y) weakly dominates the stay strategy (y, h_y). As before this means (y, f_y) has a higher chance to win but not a guaranteed one. This leads to the reduced game matrix from Table 6 to Table 7, without changing the value of the game, using Proposition 2.

	$(1,g_1)$	$(1, g'_1)$	$(1,g''_1)$	$(2,g_2)$	$(2,g'_2)$	$(2,g''_2)$	$(3,g_3)$	$(3,g'_3)$	$(3,g''_3)$	$(4,g_4)$	$(4,g'_4)$	$(4,g''_4)$
$(1,f_1)$	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
$(2,f_2)$	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
$(3,f_3)$	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)
$(4,f_4)$	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)

Table 7: Reduced Game matrix in 4-door case

The strategies (x, g'_x) and (x, g''_x) yield the same payoff equivalence because no matter what door Monty reveals, the player is going to play the same strategy, giving us the same result. Therefore using Proposition 3 we can eliminate (x, g'_x) and (x, g''_x) from Table 7 without changing the value of the game.

Letting both μ and ν to be uniform distributions leads to $\mu(y, f_y) = \frac{1}{4}$ and $\nu(y, g_y) = \frac{1}{4}$. Thus (μ, ν) is an equilibrium and we can use Definition 5 and equation 1 to compute the expected

	(1,g_1)	(2,g_2)	(3,g_3)	(4,g_4)
(1,f_1)	(0,1)	(1,0)	(1,0)	(1,0)
(2,f_2)	(1,0)	(0,1)	(1,0)	(1,0)
(3,f_3)	(1,0)	(1,0)	(0,1)	(1,0)
(4,f_4)	(1,0)	(1,0)	(1,0)	(0,1)

Table 8: Weakly dominant payoff without MH equivalent pairs

utilities of the switch strategies for the contestant.

$$\begin{aligned}
U_C(\sigma,\nu) = U_C((1,f_1),\nu) &= \sum_{t\in S_{MH}} \nu(\tau)\mu_C((1,f_1),t) \\
&= \nu(1,g_1)u_C((1,f_1),(1,g_1)) + \nu(1,g_2)u_C((1,f_1),(1,g_2)) \\
&+ \nu(1,g_3)u_C((1,f_1),(1,g_3)) + \nu(1,g_4)u_C((1,f_1),(1,g_4)) \\
&= \frac{1}{4}\times 0 + \frac{1}{4}\times 1 + \frac{1}{4}\times 1 + \frac{1}{4}\times 1 \\
&= \frac{3}{4}
\end{aligned}$$

Likewise, the expected utility for any stay strategy (y,f_y) is equal to $\frac{3}{4}$.

As before we need the full matrix Table 6 to find the expected utility of the stay strategy $(1,h_1)$ where ν has a uniform distribution with $\nu(x,g_x)=\frac{1}{12}$. Meanwhile, μ is the distribution with $\mu(y,f_y)=\frac{1}{4}$ and $\mu(y,h_y)=0$. This leads to the expected utility

$$\begin{aligned}
U_C(\sigma,\nu) = U_C((1,h_1),\nu) &= \sum_{t\in S_{MH}} \nu(\tau)\mu_C((1,h_1),t) \\
&= \nu(1,g_1)u_C((1,h_1),(1,g_1)) + \nu(1,g_1')u_C((1,h_1),(1,g_1')) + \nu(1,g_1'')u_C((1,h_1),(1,g_1'')) \\
&+ \nu(2,g_2)u_C((1,h_1),(2,g_2)) + \nu(2,g_2')u_C((1,h_1),(2,g_2')) + \nu(2,g_2'')u_C((1,h_1),(2,g_2'')) \\
&+ \nu(3,g_3)u_C((1,h_1),(3,g_3)) + \nu(3,g_3')u_C((1,h_1),(3,g_3')) + \nu(3,g_3'')u_C((1,h_1),(3,g_3'')) \\
&+ \nu(4,g_4)u_C((1,h_1),(4,g_4)) + \nu(4,g_4')u_C((1,h_1),(4,g_4')) + \nu(4,g_4'')u_C((1,h_1),(4,g_4'')) \\
&= \frac{1}{12}\times 1 + \frac{1}{12}\times 1 + \frac{1}{12}\times 1 + \frac{1}{12}\times 0 + \frac{1}{12}\times 0 + \frac{1}{12}\times 0 \\
&+ \frac{1}{12}\times 0 + \frac{1}{12}\times 0 + \frac{1}{12}\times 0 + \frac{1}{12}\times 0 + \frac{1}{12}\times 0 + \frac{1}{12}\times 0 = \frac{1}{4}
\end{aligned}$$

Therefore, the expected utility when using the stay strategy is $\frac{1}{4}$. A comparison between the expected utility of the stay strategy against the switch strategy reveals that the contestant has a better chance of winning the car when they switch their door.

Notice here that when adding another door to the game, the probability of winning by switching increases and the probability of winning by staying decreases. We will investigate if this pattern continues when we increase to 5 doors.

3.1.3 The 5-door case

In this section, we consider a game with 5 doors available for the contestant to choose from. We continue with the assumption that the contestant is staying with their initial choice until all doors are revealed except for two. Then, they will decide on switching or staying given all the previous rounds. Notice that this game gives Monty more freedom to open doors as there are three different doors to select from (four doors if the contestant initially picks the door with the car). In Table 9, we showcase Monty's strategy table, after the car has been hidden behind a door and the contestant decides which one to reveal.

(1,g_1)	g_1(1,1) = 2	g_1(1,2) = 3 g_1(1,2) = 4 g_1(1,2) = 5	g_1(1,3) = 2 g_1(1,3) = 4 g_1(1,3) = 5	g_1(1,4) = 2 g_1(1,4) = 3 g_1(1,4) = 5	g_1(1,5) = 2 g_1(1,5) = 3 g_1(1,5) = 4

$(1,g_1')$	$g_1'(1,1) = 3$	$g_1'(1,2) = 3$ $g_1'(1,2) = 4$ $g_1'(1,2) = 5$	$g_1'(1,3) = 2$ $g_1'(1,3) = 4$ $g_1'(1,3) = 5$	$g_1'(1,4) = 2$ $g_1'(1,4) = 3$ $g_1'(1,4) = 5$	$g_1'(1,5) = 2$ $g_1'(1,5) = 3$ $g_1'(1,5) = 4$
$(1,g_1'')$	$g_1''(1,1) = 4$	$g_1''(1,2) = 3$ $g_1''(1,2) = 4$ $g_1''(1,2) = 5$	$g_1''(1,3) = 2$ $g_1''(1,3) = 4$ $g_1''(1,3) = 5$	$g_1''(1,4) = 2$ $g_1''(1,4) = 3$ $g_1''(1,4) = 5$	$g_1''(1,5) = 2$ $g_1''(1,5) = 3$ $g_1''(1,5) = 4$
$(1,g_1''')$	$g_1'''(1,1) = 5$	$g_1'''(1,2) = 3$ $g_1'''(1,2) = 4$ $g_1'''(1,2) = 5$	$g_1'''(1,3) = 2$ $g_1'''(1,3) = 4$ $g_1'''(1,3) = 5$	$g_1'''(1,4) = 2$ $g_1'''(1,4) = 3$ $g_1'''(1,4) = 5$	$g_1'''(1,5) = 2$ $g_1'''(1,5) = 3$ $g_1'''(1,5) = 4$
$(2,g_2)$	$g_2(2,2) = 1$	$g_2(2,1) = 3$ $g_2(2,1) = 4$ $g_2(2,1) = 5$	$g_2(2,3) = 1$ $g_2(2,3) = 4$ $g_2(2,3) = 5$	$g_2(2,4) = 1$ $g_2(2,4) = 3$ $g_2(2,4) = 5$	$g_2(2,5) = 1$ $g_2(2,5) = 3$ $g_2(2,5) = 4$
$(2,g_2')$	$g_2'(2,2) = 3$	$g_2'(2,1) = 3$ $g_2'(2,1) = 4$ $g_2'(2,1) = 5$	$g_2'(2,3) = 1$ $g_2'(2,3) = 4$ $g_2'(2,3) = 5$	$g_2'(2,4) = 1$ $g_2'(2,4) = 3$ $g_2'(2,4) = 5$	$g_2'(2,5) = 1$ $g_2'(2,5) = 3$ $g_2'(2,5) = 4$
$(2,g_2'')$	$g_2''(2,2) = 4$	$g_2''(2,1) = 3$ $g_2''(2,1) = 4$ $g_2''(2,1) = 5$	$g_2''(2,3) = 1$ $g_2''(2,3) = 4$ $g_2''(2,3) = 5$	$g_2''(2,4) = 1$ $g_2''(2,4) = 3$ $g_2''(2,4) = 5$	$g_2''(2,5) = 1$ $g_2''(2,5) = 3$ $g_2''(2,5) = 4$
$(2,g_2''')$	$g_2'''(2,2) = 5$	$g_2'''(2,1) = 3$ $g_2'''(2,1) = 4$ $g_2'''(2,1) = 5$	$g_2'''(2,3) = 1$ $g_2'''(2,3) = 4$ $g_2'''(2,3) = 5$	$g_2'''(2,4) = 1$ $g_2'''(2,4) = 3$ $g_2'''(2,4) = 5$	$g_2'''(2,5) = 1$ $g_2'''(2,5) = 3$ $g_2'''(2,5) = 4$
$(3,g_3)$	$g_3(3,3) = 1$	$g_3(3,1) = 2$ $g_3(3,1) = 4$ $g_3(3,1) = 5$	$g_3(3,2) = 1$ $g_3(3,2) = 4$ $g_3(3,2) = 5$	$g_3(3,4) = 1$ $g_3(3,4) = 2$ $g_3(3,4) = 5$	$g_3(3,5) = 1$ $g_3(3,5) = 2$ $g_3(3,5) = 4$
$(3,g_3')$	$g_3'(3,3) = 2$	$g_3'(3,1) = 2$ $g_3'(3,1) = 4$ $g_3'(3,1) = 5$	$g_3'(3,2) = 1$ $g_3'(3,2) = 4$ $g_3'(3,2) = 5$	$g_3'(3,4) = 1$ $g_3'(3,4) = 2$ $g_3'(3,4) = 5$	$g_3'(3,5) = 1$ $g_3'(3,5) = 2$ $g_3'(3,5) = 4$
$(3,g_3'')$	$g_3''(3,3) = 4$	$g_3''(3,1) = 2$ $g_3''(3,1) = 4$ $g_3''(3,1) = 5$	$g_3''(3,2) = 1$ $g_3''(3,2) = 4$ $g_3''(3,2) = 5$	$g_3''(3,4) = 1$ $g_3''(3,4) = 2$ $g_3''(3,4) = 5$	$g_3''(3,5) = 1$ $g_3''(3,5) = 2$ $g_3''(3,5) = 4$
$(3,g_3''')$	$g_3'''(3,3) = 5$	$g_3'''(3,1) = 2$ $g_3'''(3,1) = 4$ $g_3'''(3,1) = 5$	$g_3(3,2) = 1$ $g_3'''(3,2) = 4$ $g_3'''(3,2) = 5$	$g_3(3,4) = 1$ $g_3'''(3,4) = 2$ $g_3'''(3,4) = 5$	$g_3'''(3,5) = 1$ $g_3'''(3,5) = 2$ $g_3'''(3,5) = 4$
$(4,g_4)$	$g_4(4,4) = 1$	$g_4(4,1) = 2$ $g_4(4,1) = 3$ $g_4(4,1) = 5$	$g_4(4,2) = 1$ $g_4(4,2) = 3$ $g_4(4,2) = 5$	$g_4(4,3) = 1$ $g_4(4,3) = 2$ $g_4(4,3) = 5$	$g_4(4,5) = 1$ $g_4(4,5) = 2$ $g_4(4,5) = 3$
$(4,g_4')$	$g_4'(4,4) = 2$	$g_4'(4,1) = 2$ $g_4'(4,1) = 3$ $g_4'(4,1) = 5$	$g_4'(4,2) = 1$ $g_4'(4,2) = 3$ $g_4'(4,2) = 5$	$g_4'(4,3) = 1$ $g_4'(4,3) = 2$ $g_4'(4,3) = 5$	$g_4'(4,5) = 1$ $g_4'(4,5) = 2$ $g_4'(4,5) = 3$
$(4,g_4'')$	$g_4''(4,4) = 3$	$g_4''(4,1) = 2$ $g_4''(4,1) = 3$ $g_4''(4,1) = 5$	$g_4''(4,2) = 1$ $g_4''(4,2) = 3$ $g_4''(4,2) = 5$	$g_4''(4,3) = 1$ $g_4''(4,3) = 2$ $g_4''(4,3) = 5$	$g_4''(4,5) = 1$ $g_4''(4,5) = 2$ $g_4''(4,5) = 3$
$(4,g_4''')$	$g_4'''(4,4) = 5$	$g_4'''(4,1) = 2$ $g_4'''(4,1) = 3$ $g_4'''(4,1) = 5$	$g_4'''(4,2) = 1$ $g_4'''(4,2) = 3$ $g_4'''(4,2) = 5$	$g_4'''(4,3) = 1$ $g_4'''(4,3) = 2$ $g_4'''(4,3) = 5$	$g_4'''(4,5) = 1$ $g_4'''(4,5) = 2$ $g_4'''(4,5) = 3$
$(5,g_5)$	$g_5(5,5) = 1$	$g_5(5,1) = 2$ $g_5(5,1) = 3$ $g_5(5,1) = 4$	$g_5(5,2) = 1$ $g_5(5,2) = 3$ $g_5(5,2) = 4$	$g_5(5,3) = 1$ $g_5(5,3) = 2$ $g_5(5,3) = 4$	$g_5(5,4) = 1$ $g_5(5,4) = 2$ $g_5(5,4) = 3$
$(5,g_5')$	$g_5'(5,5) = 2$	$g_5'(5,1) = 2$ $g_5'(5,1) = 3$ $g_5'(5,1) = 4$	$g_5'(5,2) = 1$ $g_5'(5,2) = 3$ $g_5'(5,2) = 4$	$g_5'(5,3) = 1$ $g_5'(5,3) = 2$ $g_5'(5,3) = 4$	$g_5'(5,4) = 1$ $g_5'(5,4) = 2$ $g_5'(5,4) = 3$
$(5,g_5'')$	$g_5''(5,5) = 3$	$g_5''(5,1) = 2$ $g_5''(5,1) = 3$	$g_5''(5,2) = 1$ $g_5''(5,2) = 3$	$g_5''(5,3) = 1$ $g_5''(5,3) = 2$	$g_5''(5,4) = 1$ $g_5''(5,4) = 2$

		$g''_5(5,1) = 4$	$g''_5(5,2) = 4$	$g''_5(5,3) = 4$	$g''_5(5,4) = 3$
$(5,g'''_5)$	$g'''_5(5,5) = 4$	$g'''_5(5,1) = 2$ $g'''_5(5,1) = 3$ $g'''_5(5,1) = 4$	$g'''_5(5,2) = 1$ $g'''_5(5,2) = 3$ $g'''_5(5,2) = 4$	$g'''_5(5,3) = 1$ $g'''_5(5,3) = 2$ $g'''_5(5,3) = 4$	$g'''_5(5,4) = 1$ $g'''_5(5,4) = 2$ $g'''_5(5,4) = 3$

Table 9: MH strategy in 5-door game

With the number of doors increased, it might be hard to imagine the contestant having a better chance at winning the car. Thus, we take a look at the payoff for both the contestant and Monty in Table 10. As a reminder, the first column outlines all of the contestant's strategies, (y, h_y) and (y, f_y), while the first row outlines all of Monty's strategies (x, g_x).

	$(1,g_1)$	$(1, g'_1)$	$(1,g''_1)$	$(1, g'''_1)$	$(2,g_2)$	$(2,g'_2)$	$(2,g''_2)$	$(2,g'''_2)$	$(3,g_3)$	$(3,g'_3)$	$(3,g''_3)$	$(3,g'''_3)$	$(4,g_4)$	$(4,g'_4)$	$(4,g''_4)$	$(4,g'''_4)$	$(5,g_5)$	$(5,g'_5)$	$(5,g''_5)$	$(5,g'''_5)$
$(1,h_1)$	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)
$(2,h_2)$	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)
$(3,h_3)$	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)
$(4,h_4)$	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)
$(5,h_5)$	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)
$(1,f_1)$	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
$(2,f_2)$	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
$(3,f_3)$	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
$(4,f_4)$	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)
$(5,f_5)$	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)

Table 10: Payoff Matrix in 5-door game

For the event that corresponds to the cell with $(4, g''_4)$ and $(2, f_2)$, Monty hid the prize behind Door 4 and the contestant initially chose Door 2. Therefore, Monty has the option to either open Doors 1, 3, or 5 while the contestant stays with Door 2 until Monty reveals all three doors except doors 2 and 4 (the prize door). The intersection cell of $(4, g''_4)$ and $(2, f_2)$ also indicates that the contestant switched to Door 4 in the last round, resulting in a win. From Table 10, each row in the stay strategy h_y gives the contestant $\frac{4}{20}$ or $\frac{1}{5}$ chance to win the car. In contrast, the switch strategy f_y gives the contestant a significantly higher probability, $\frac{4}{5}$, to win the car.

Likewise the previous cases, the stay strategy, (y, h_y), is still weakly dominated by the switch strategy, (y, f_y), because the contestant has a higher probability of winning if they switch rather than stay without a guarantee of winning every time. For example, if the contestant initially chose the door with the prize, the switch strategy would result in a loss. As a result, using Proposition 2 eliminates all the stay strategies, (y, h_y), without changing the value of the game. Table 11 summarizes the reduced payoff matrix.

	$(1,g_1)$	$(1, g'_1)$	$(1,g''_1)$	$(1, g'''_1)$	$(2,g_2)$	$(2,g'_2)$	$(2,g''_2)$	$(2,g'''_2)$	$(3,g_3)$	$(3,g'_3)$	$(3,g''_3)$	$(3,g'''_3)$	$(4,g_4)$	$(4,g'_4)$	$(4,g''_4)$	$(4,g'''_4)$	$(5,g_5)$	$(5,g'_5)$	$(5,g''_5)$	$(5,g'''_5)$
$(1,f_1)$	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
$(2,f_2)$	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
$(3,f_3)$	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)
$(4,f_4)$	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)
$(5,f_5)$	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)

Table 11: Payoff of the weakly dominant strategy in 5-door game

Further reduction can be done in the payoff matrix since the strategies (x, g'_x), (x, g''_x), and (x, g'''_x) yield the same payoff. This reduction is due to Proposition 3 and is summarized in Table 12.

	$(1,g_1)$	$(2,g_2)$	$(3,g_3)$	$(4,g_4)$	$(5,g_5)$
$(1,f_1)$	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)
$(2,f_2)$	(1,0)	(0,1)	(1,0)	(1,0)	(1,0)
$(3,f_3)$	(1,0)	(1,0)	(0,1)	(1,0)	(1,0)
$(4,f_4)$	(1,0)	(1,0)	(1,0)	(0,1)	(1,0)
$(5,f_5)$	(1,0)	(1,0)	(1,0)	(1,0)	(0,1)

Table 12: Payoff of the weakly dominant strategy without MH equivalent pairs

Assuming that both μ and ν following a uniform distribution with $\mu(1, f_y) = \frac{1}{5}$ and $\nu(y, g_y) = \frac{1}{5}$, leads to the pair (μ, ν) being in an equilibrium. Therefore, we use Table 12, Definition 5 and equation 1 to calculate the expected utility of the switch strategy.

$$\begin{aligned}U_C(\sigma,\nu) = U_C((1,f_1),\nu) &= \sum_{t\in S_{MH}} \nu(\tau)\mu_C((1,f_1),t)\\
&= \nu(1,g_1)u_C((1,f_1),(1,g_1)) + \nu(1,g_2)u_C((1,f_1),(1,g_2))\\
&+ \nu(1,g_3)u_C((1,f_1),(1,g_3)) + \nu(1,g_4)u_C((1,f_1),(1,g_4))\\
&+ \nu(1,g_5)u_C((1,f_1),(1,g_5))\\
&= \frac{1}{5}\times 0 + \frac{1}{5}\times 1 + \frac{1}{5}\times 1 + \frac{1}{5}\times 1 + \frac{1}{5}\times 1\\
&= \frac{4}{5}.\end{aligned}$$

For the stay strategy, we need to use the full payoff matrix, Table 10, Definition 5, and equation 1. Additionally, we assume that ν follows a uniform distribution with probability $\nu(x,g_x) = \frac{1}{20}$ while μ is the distribution with probabilities $\mu(y,f_y) = \frac{1}{20}$ and $\mu(y,h_y) = 0$. Thus, the expected utility for the strategy $(1,h_1)$ is given by

$$\begin{aligned}U_C(\sigma,\nu) = U_C((1,h_1),\nu) &= \sum_{t\in S_{MH}} \nu(\tau)\mu_C((1,h_1),t)\\
&= \nu(1,g_1)u_C((1,h_1),(1,g_1)) + \nu(1,g_1')u_C((1,h_1),(1,g_1'))\\
&+ \nu(1,g_1'')u_C((1,h_1),(1,g_1'')) + \nu(1,g_1''')u_C((1,h_1),(1,g_1'''))\\
&+ \nu(2,g_2)u_C((1,h_1),(2,g_2)) + \nu(2,g_2')u_C((1,h_1),(2,g_2'))\\
&+ \nu(2,g_2'')u_C((1,h_1),(2,g_2'')) + \nu(2,g_2''')u_C((1,h_1),(2,g_2'''))\\
&+ \nu(3,g_3)u_C((1,h_1),(3,g_3)) + \nu(3,g_3')u_C((1,h_1),(3,g_3'))\\
&+ \nu(3,g_3'')u_C((1,h_1),(3,g_3'')) + \nu(3,g_3''')u_C((1,h_1),(3,g_3'''))\\
&+ \nu(4,g_4)u_C((1,h_1),(4,g_4)) + \nu(4,g_4')u_C((1,h_1),(4,g_4'))\\
&+ \nu(4,g_4'')u_C((1,h_1),(4,g_4'')) + \nu(4,g_4''')u_C((1,h_1),(4,g_4'''))\\
&+ \nu(5,g_5)u_C((1,h_1),(5,g_5)) + \nu(5,g_5')u_C((1,h_1),(5,g_5'))\\
&+ \nu(5,g_5'')u_C((1,h_1),(5,g_5'')) + \nu(5,g_5''')u_C((1,h_1),(5,g_5'''))\\
&= \frac{1}{20}\times 1 + \frac{1}{20}\times 1 + \frac{1}{20}\times 1 + \frac{1}{20}\times 1 + \frac{1}{20}\times 0 + \frac{1}{20}\times 0\\
&+ \frac{1}{20}\times 0 + \frac{1}{20}\times 0 + \frac{1}{20}\times 0 + \frac{1}{20}\times 0 + \frac{1}{20}\times 0 + \frac{1}{20}\times 0\\
&+ \frac{1}{20}\times 0 + \frac{1}{20}\times 0 + \frac{1}{20}\times 0 + \frac{1}{20}\times 0 + \frac{1}{20}\times 0 + \frac{1}{20}\times 0\\
&+ \frac{1}{20}\times 0 + \frac{1}{20}\times 0\\
&= \frac{1}{5}.\end{aligned}$$

Similarly, all the other rows for the stay strategy results in equal expected utility. We continue to see that both game theory and the probability approaches confirm the contestant has a higher chance of winning the prize of the game when relying on the switching strategy. In addition, the trend of increasing the probability of winning of the switch strategy continues as we add more doors to the game while the stay strategy results in decreasing the contestant's chance of winning.

3.2 The n^{th}-door case

Before considering the general case of having n-doors in the game, we summarize and compare the results of both probability and game theoretic approaches.

For the 3-door case, Bayesian and game theoretic approaches agree that the optimal strategy of winning for the contestant is by switching their initial choice. This strategy has a winning probability of $\frac{2}{3}$. Additionally, the contestant has a probability of $\frac{3}{4}$ to win when they follow the switch strategy during the 4-door case. Furthermore, in the 5-door case, the switch strategy wins with a probability of $\frac{4}{5}$.

Next, we assume that there are n doors in the game for the contestant to choose from initially with each having equal probability of $\frac{1}{n}$ to have the prize hidden behind it. In other words, $P(D_i) = \frac{1}{n}$ for $i = 1, 2, \cdots, n$.

Once again, suppose that the car is hidden behind Door 1 and the contestant initially selected Door 3. Thus, the probability of Monty uncovering Door 2 is $P(B|D_1) = \frac{1}{n-2}$ as Monty can not display the door with the prize nor the contestant's choice. Similarly, $P(B|D_i) = \frac{1}{n-2}$ for all i except $i = 2$ or 3. For $i = 2$, the probability $P(B|D_2) = 0$ as Monty can not reveal Door 2 because it has the prize. Meanwhile, $P(B|D_3) = \frac{1}{n-1}$ as this the probability of Monty opening Door 2 when the contestant chose Door 2 and it hides the car behind it. Therefore, the overall probability of Monty choosing Door 2 is given by:

$$\begin{aligned}
P(B) &= \sum_{i=1}^{n} P(D_i)P(B|D_i) \\
&= P(D_2)P(B|D_2) + P(D_3)P(B|D_3) + P(D_1)P(B|D_1) + \cdots + P(D_n)P(B|D_n) \\
&= 0 + \frac{1}{n}\frac{1}{n-1} + \underbrace{\frac{1}{n}\frac{1}{n-2} + \cdots + \frac{1}{n}\frac{1}{n-2}}_{n-2} \\
&= \frac{1}{n-1}.
\end{aligned}$$

Next, the conditional probabilities of the prize being hidden in the back of Door i knowing that Monty has chosen to unveil Door 2 is as follows:

$$P(D_2|B) = \frac{P(D_2)P(B|D_2)}{P(B)} = \frac{1/n \times 0}{1/(n-1)} = 0$$

$$P(D_3|B) = \frac{P(D_3)P(B|D_3)}{P(B)} = \frac{1/n \times 1/(n-1)}{1/(n-1)} = \frac{1}{n}$$

and for all other i,

$$P(D_i|B) = \frac{P(D_i)P(B|D_i)}{P(B)} = \frac{1/n \times 1/(n-2)}{1/(n-1)} = \frac{n-1}{n(n-2)}.$$

Therefore, the probability of winning the car when the contestant decide to switch their initial choice of Door 3 to the prize bearing Door 2 is $\sum_{\substack{i=1 \\ i\neq 2,3}}^{n} P(D_i|B) = (n-2) \times \frac{n-1}{n(n-2)} = \frac{n-1}{n}$.

Next, we showcase the game theoretic approach when there are n doors that the contestant can choose from. The payoff matrix with both Monty and the contestant strategies would look like Table 13. In addition, the reduced payoff table is given by Table 14.

	$(1,g_1)$	$(1, g_1')$	$\cdots$	$(1, g_1^{(n-2)th})$	$(2,g_2)$	$(2,g_2')$	$\cdots$	$(2,g_2^{(n-2)th})$	$(3,g_3)$	$(3,g_3')$	$\cdots$	$(3,g_3^{(n-2)th})$	$\cdots$	(n, g_n)	(n,g_n')	$\cdots$	$(n,g_n^{(n-2)th})$
$(1,h_1)$	(1,0)	(1,0)	$\cdots$	(1,0)	(0,1)	(0,1)	$\cdots$	(0,1)	(0,1)	(0,1)	$\cdots$	(0,1)	$\cdots$	(0,1)	(0,1)	$\cdots$	(0,1)
$(2,h_2)$	(0,1)	(0,1)	$\cdots$	(0,1)	(1,0)	(1,0)	$\cdots$	(1,0)	(0,1)	(0,1)	$\cdots$	(0,1)	$\cdots$	(0,1)	(0,1)	$\cdots$	(0,1)
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
(n,h_n)	(0,1)	(0,1)	$\cdots$	(0,1)	(0,1)	(0,1)	$\cdots$	(0,1)	(0,1)	(0,1)	$\cdots$	(0,1)	$\cdots$	(1,0)	(1,0)	$\cdots$	(1,0)
$(1,f_1)$	(0,1)	(0,1)	$\cdots$	(0,1)	(1,0)	(1,0)	$\cdots$	(1,0)	(1,0)	(1,0)	$\cdots$	(1,0)	$\cdots$	(1,0)	(1,0)	$\cdots$	(1,0)
$(2,f_2)$	(1,0)	(1,0)	$\cdots$	(1,0)	(0,1)	(0,1)	$\cdots$	(0,1)	(1,0)	(1,0)	$\cdots$	(1,0)	$\cdots$	(1,0)	(1,0)	$\cdots$	(1,0)
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
(n,f_n)	(1,0)	(1,0)	$\cdots$	(1,0)	(1,0)	(1,0)	$\cdots$	(1,0)	(1,0)	(1,0)	$\cdots$	(1,0)	$\cdots$	(0,1)	(0,1)	$\cdots$	(0,1)

Table 13: Payoff Matrix in n-door game

	(1,g_1)	(2,g_2)	(3,g_3)	$\cdots$	(n, g_n)
(1,f_1)	(0,1)	(1,0)	(1,0)	$\cdots$	(1,0)
(2,f_2)	(1,0)	(0,1)	(1,0)	$\cdots$	(1,0)
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
(n,f_n)	(1,0)	(1,0)	(1,0)	$\cdots$	(0,1)

Table 14: Reduced Payoff Matrix in n-door game

Under uniform distribution for both μ and ν with probabilities of $\mu(1, f_y) = \frac{1}{n}$ and $\nu(y, g_y) = \frac{1}{n}$, the expected utility of the switch strategy is given by:

$$\begin{aligned} U_C(\sigma, \nu) = U_C((1, f_1), \nu) &= \sum_{t \in S_{MH}} \nu(\tau)\mu_C((1, f_1), t) \\ &= \nu(1, g_1)u_C((1, f_1), (1, g_1)) + \nu(1, g_2)u_C((1, f_1), (1, g_2)) \\ &+ \nu(1, g_3)u_C((1, f_1), (1, g_3)) + \cdots + \nu(1, g_n)u_C((1, f_1), (1, g_n)) \\ &= \frac{1}{n} \times 0 + \underbrace{\frac{1}{n} \times 1 + \frac{1}{n} \times 1 + \cdots + \frac{1}{n} \times 1}_{(n-1)-\, terms} \\ &= \frac{n-1}{n}. \end{aligned}$$

However, using Table 13 and the assumption that ν has a uniform distribution with $\nu(x, g_x) = \frac{1}{n}$ and μ being a distribution with probabilities of $\mu(y, h_y) = 0$ and $\mu(y, f_y) = \frac{1}{n}$. Then, the expected utility of the stay strategy $(1, h_1)$ is given by:

$$\begin{aligned} U_C(\sigma, \nu) = U_C((1, h_1), \nu) &= \sum_{t \in S_{MH}} \nu(\tau)\mu_C((1, h_1), t) \\ &= \nu(1, g_1)u_C((1, h_1), (1, g_1)) + \nu(1, g_1')u_C((1, h_1), (1, g_1')) \\ &+ \cdots + \nu(1, g_1^{(n-2)th})u_C((1, h_1), (1, g_1^{(n-2)th})) \\ &+ \nu(2, g_2)u_C((1, h_1), (2, g_2)) + \nu(2, g_2')u_C((1, h_1), (2, g_2')) \\ &+ \cdots + \nu(2, g_2^{(n-2)th})u_C((1, h_1), (2, g_2^{(n-2)th})) \\ &+ \nu(3, g_3)u_C((1, h_1), (3, g_3)) + \nu(3, g_3')u_C((1, h_1), (3, g_3')) \\ &+ \cdots + \nu(3, g_3^{(n-2)th})u_C((1, h_1), (3, g_3^{(n-2)th})) \\ &+ \cdots \\ &+ \nu(n, g_n)u_C((1, h_1), (n, g_n)) + \nu(n, g_n')u_C((1, h_1), (n, g_n')) \\ &+ \cdots + \nu(n, g_n^{(n-2)th})u_C((1, h_1), (n, g_n^{(n-2)th})) \\ &= \underbrace{\frac{1}{n(n-1)} \times 1 + \frac{1}{n(n-1)} \times 1 + \cdots + \frac{1}{n(n-1)} \times 1}_{n-1} \\ &+ \frac{1}{n(n-1)} \times 0 + \frac{1}{n(n-1)} \times 0 + \cdots + \frac{1}{n(n-1)} \times 0 \\ &+ \cdots + \frac{1}{n(n-1)} \times 0 + \frac{1}{n(n-1)} \times 0 + \cdots + \frac{1}{n(n-1)} \times 0 \\ &= \frac{1}{n}. \end{aligned}$$

As a conclusion, both Bayes' theorem and the game theoretic approach favored the switch strategy for the contestant with a probability of $\frac{n-1}{n}$ to win the prize.

4 Two Players Monty

4.1 The 3-door case

The authors in [2] describe a new version of Monty Hall game. In this section, we assume there are two contestants in the game in addition to the following rules:

- Each contestant is aware that there is another player in the game other than Monty, but has no knowledge of what door they selected initially.
- When the contestants pick the same door, the classic version of Monty game proceeds.
- In the last round, if the contestant choose different doors, Monty has to reveal the remaining door, regardless of what is hidden behind it.

Since there are three doors in the game, we will attempt to capture the efficacy of the stay/switch strategies for each contestant.

We let h_j^1 and f_j^1 to represent the stay and switch strategies respectively for Contestant 1. Meanwhile, we denote these strategies with h_j^2 and f_j^2 for Contestant 2. As a reminder, switching occurs when there are only two doors left in the game. Next, suppose there are three doors in the game and that the payoff pairs for each player are given as follows:

- (0,1) will represent a win for Contestant 1.
- (1,0) will represent a win for Contestant 2.
- (1,1) will represent a win for both Contestants 1 and 2.
- (0,0) will represent a loss for both Contestant 1 and 2.
- X means Monty had no choice but to reveal the door with the car, so the game is over. This will not count as a win/loss for the contestants since the game was not able to proceed using the stay/switch strategies.

Suppose that Monty hides the prize in the back of Door 1. Table 15 presents the payoff pairs of the stay/switch strategies for both contestants.

	$(1,h_1^1)$	$(2,h_2^1)$	$(3,h_3^1)$	$(1,f_1^1)$	$(2,f_2^1)$	$(3,f_3^1)$
$(1,h_1^2)$	(1,1)	(1,0)	(1,0)	(1,0)	(1,1)	(1,1)
$(2,h_2^2)$	(0,1)	(0,0)	X	(0,0)	(0,1)	X
$(3,h_3^2)$	(0,1)	X	(0,0)	(0,0)	X	(1,0)
$(1,f_1^2)$	(0,1)	(0,0)	(0,0)	(0,0)	(0,1)	(0,1)
$(2,f_2^2)$	(1,1)	(1,0)	X	(1,1)	(1,1)	X
$(3,f_3^2)$	(1,1)	X	(1,0)	(1,0)	X	(1,1)

Table 15: 3-door Payoff Table of Contestants 1 and 2

Note that the intersection cell of $(1, h_1^1)$ and $(1, h_1^2)$ represent the event that both Contestants 1 and 2 initially chose Door 1 and decided to stay with their choice even after Monty has opened Door 2. Thus, both contestants win the game. On the other hand, the intersection cell of $(1, f_1^1)$ and $(1, h_1^2)$ reveals the event that both contestants initially chose Door 1, however, Contestant 1 chose to switch to Door 3 after Monty opened door 2. Hence, this sequence of events yields a win for Contestant 2. Furthermore, the game will end when Contestant 1 chooses Door 2 while Contestant 2 chooses Door 3, forcing an X in the intersection cell of $(2, f_1^1)$ and $(3, h_3^2)$ in the payoff table.

We will disregard the cells with quantity X when calculating the probabilities of the switch and stay strategies for each contestant. This is due the fact that the contestants were unable to use their stay/switch strategies to play the game and hence they will not count for/against them.

In this 3-door game, the first three rows in Table 15 consists of all the possible scenarios when Contestant 2 follows the stay strategy while the first three columns represent the payoff of Contestant 2 under the same strategy. On the other hand, the last three rows show the payoff pairs of the switch strategy for Contestant 2 and the last three columns are the payoff for Contestant 1.

To compute the overall win for Contestant 1 under the stay strategy, we count all the pairs with 1 in the second place of the pair and divide by the number of all possibilities disregarding the cells with X. Therefore, Contestant 1 will win the car $\frac{6}{14} = \frac{3}{7}$ of the time. In contrast, following the switch strategy gives Contestant 1 a chance of $\frac{8}{14} = \frac{4}{7}$ win the car. Likewise, Contestant 2 has the same probabilities as Contestant 1 to win the car under the two strategies.

With the added player, we clearly notice that the probability of winning the car decreases from $\frac{2}{3} = \frac{14}{21}$ to $\frac{4}{7} = \frac{12}{21}$ following the switch strategy while it increases from $\frac{1}{3} = \frac{7}{21}$ to $\frac{3}{7} = \frac{9}{21}$ under the staying strategy. Hence, the difference between the probabilities to win following either the switching or staying strategies is getting smaller.

A payoff table similar to Table 15 can be constructed when Monty hides the prize behind either Doors 2 or 3. This will also lead to similar probabilities of the stay and switch strategies for both contestants.

4.2 The 4-door case

The rules remain the same for the 4-door case in the two-player game. We are still assuming that Monty hides the prize behind Door 1. Recall that since the number of doors $n > 3$, the "switch" strategy means that each contestant will stay with their initial choice until there are two doors left, in which case they will switch to the other unopened door.

In Table 16, the first row outlines all of the stay/switch strategies for Contestant 1 and the first column lists all of Contestant 2 strategies. As before, $(0, 1)$ denotes a win for Contestant 1 and $(1, 0)$ represents a win for Contestant 2. Additionally, $(1, 1)$ is the scenario when both contestants win while $(0, 0)$ is when both lose.

	$(1,h_1^1)$	$(2,h_2^1)$	$(3,h_3^1)$	$(4,h_4^1)$	$(1,f_1^1)$	$(2,f_2^1)$	$(3,f_3^1)$	$(4,f_4^1)$
$(1,h_1^2)$	(1,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,1)	(1,1)	(1,1)
$(2,h_2^2)$	(0,1)	(0,0)	X	X	(0,0)	(0,1)	X	X
$(3,h_3^2)$	(0,1)	X	(0,0)	X	(0,0)	X	(0,1)	X
$(4,h_4^2)$	(0,1)	X	X	(0,0)	(0,0)	X	X	(0,1)
$(1,f_1^2)$	(0,1)	(0,0)	(0,0)	(0,0)	(0,0)	(0,1)	(0,1)	(0,1)
$(2,f_2^2)$	(1,1)	(1,0)	X	X	(1,0)	(1,1)	X	X
$(3,f_3^2)$	(1,1)	X	(1,0)	X	(1,0)	X	(1,1)	X
$(4,f_4^2)$	(1,1)	X	X	(1,0)	(1,0)	X	X	(1,1)

Table 16: 4-door Payoff Table of Contestants 1 and 2

We continue to use the quantity X to denote scenarios where Monty had no choice but to reveal the prize, ending the game before the contestants can utilize their strategies.

Using the stay strategy, each contestant will win $\frac{8}{20} = \frac{2}{5}$ of the time, whereas using the switch strategy, they will win $\frac{12}{20} = \frac{3}{5}$ of the time. This is interesting because we can see that the contestants' chances of winning are slightly increased, from $\frac{4}{7}$ to $\frac{3}{5}$ under the optimal strategy of switching, when we add another door to the game. Also, compared with the classic-Monty, the probability of winning for the player has decreased when adding a new contestant.

4.3 The 5-door case

In this section, we present the payoff table for each contestant under the same rules and conditions we considered in the 3-door case.

	$(1,h^1_1)$	$(2,h^1_2)$	$(3,h^1_3)$	$(4,h^1_4)$	$(5,h^1_5)$	$(1,f^1_1)$	$(2,f^1_2)$	$(3,f^1_3)$	$(4,f^1_4)$	$(5,f^1_5)$
$(1,h^2_1)$	(1,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,1)	(1,1)	(1,1)	(1,1)
$(2,h^2_2)$	(0,1)	(0,0)	X	X	X	(0,0)	(1,0)	X	X	X
$(3,h^2_3)$	(0,1)	X	(0,0)	X	X	(0,0)	X	(0,1)	X	X
$(4,h^2_4)$	(0,1)	X	X	(0,0)	X	(0,0)	X	X	(0,1)	X
$(5,h^2_5)$	(0,1)	X	X	X	(0,0)	(0,0)	X	X	X	(0,1)
$(1,f^2_1)$	(0,1)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,1)	(0,1)	(0,1)	(0,1)
$(2,f^2_2)$	(1,1)	(1,0)	X	X	X	(1,0)	(1,1)	X	X	X
$(3,f^2_3)$	(1,1)	X	(1,0)	X	X	(1,0)	X	(1,1)	X	X
$(4,f^2_4)$	(1,1)	X	X	(1,0)	X	(1,0)	X	X	(1,1)	X
$(5,f^2_5)$	(1,1)	X	X	X	(1,0)	(1,0)	X	X	X	(1,1)

Table 17: 4-door Payoff Table of Contestants 1 and 2

Notice how there appear to be many more X's than in the previous tables. That is because the game can only continue with the contestants using their strategies when at least one of them has initially chosen the door with the prize, or if they both initially chose the same door. When they choose their own separate doors from the prize, this is when three doors remain and Monty has no choice but to open the door with the prize, ultimately ending the game.
When we calculate the probability of winning, we do not consider the scenarios denoted with an X, since the contestants didn't have a chance to use their stratgies before the game had ended. In the 5-door case, when using the stay strategy, each contestant will win $\frac{10}{26} = \frac{5}{13}$ of the time. When using the switch strategy, the contestant will win $\frac{16}{26} = \frac{8}{13}$ of the time. $\frac{8}{15}$ is still slightly higher than the probabilities of the switch strategy in the 3- and 4-door cases, but still lower than the 5-door case in the normal Monty case.

There exists an interesting relationship when comparing the probability fractions between the different cases. Comparing these probabilities informs us if the chances of winning increase or decrease as we add doors to the game. In order to compare these fractions, we need to find its least common denominator. Let's start with the 3-door and 4-door case. The least common denominator between these two probabilities is 35.

	3-door case	4-door case
Stay strategy	$\frac{3}{7} = \frac{15}{35}$	$\frac{2}{5} = \frac{14}{35}$
Switch strategy	$\frac{4}{7} = \frac{20}{35}$	$\frac{3}{5} = \frac{21}{35}$

Table 18: Probability Comparisons between 3-door and 4-door Cases

Notice that the probabilities of the stay strategies between these two cases only differ by one fractional unit. The same is true for the probabilities of the switch strategies.

Now let's look at the probabilities of the strategies between the 4-door and 5-door cases. the least common denominator here is 65.

	4-door case	5-door case
Stay strategy	$\frac{2}{5} = \frac{26}{65}$	$\frac{5}{13} = \frac{25}{65}$
Switch strategy	$\frac{3}{5} = \frac{39}{65}$	$\frac{8}{13} = \frac{40}{65}$

Table 19: Probability Comparisons between 4-door and 5-door Cases

The relationship between the two cases still remains, where the probabilities of the stay strategy between the 4-door and 5-door cases only differ by a fractional unit.

4.4 Additional Versions of the Game

The authors in [3] describe several other versions of the Monty Hall game and the new probabilities that would arise from the new game and its new set of rules. In this paper, they used the variables C_i = the car is hidden behind Door i, S_i = the contestant initially select Door i, and H_i = Monty opens Door i after the contestant made the initial choice. Monty still cannot open the door with the prize nor the door that the contestant chose. The paper outlines how the probabilities of the contestant winning changes between the different versions of the game. Without loss of generality, we are assuming that the contestant initially chose Door 1, and that Door 3 has been opened.

4.4.1 A Malevolent Host

Monty's strategy is to offer the contestant to switch their door only if the initial choice was correct. The contestant is unaware of this strategy. In this scenario, it's advised that the contestant should stick with Door 1.

4.4.2 A Decision Based on Past Experience

Let's say that the contestant has spent years observing the game show and taking notes of how often the prize is hidden behind each of the doors, as well as Monty's different strategies and behaviors. Suppose they observed that Monty hides the prize behind Door 1 50% of the time, behind Door 2 35% of the time, and behind Door 3 15% of the time. Also, once in every 10 games Monty will end the game by saying either "Well done!" if the contestant initially chose the door with the prize, or "Bad luck!" if the contestant's initial choice was incorrect. In addition, the contestant is able to guess where the car is 60% of the time solely by observing Monty's body language. With these observations, the authors calculated the chances of winning by switching to Door 2 to be only 31.8%. Thus, the contestant should stick with Door 1.

4.4.3 A Host who Prefers to Open Doors on the Right

Monty has a new rule of always opening the rightmost door that conceals a goat. If the contestant was made aware of this rule prior to playing, they would have a much easier time figuring out which door hides the prize: if there are four doors and Monty opens Door 3, then the contestant would know immediately that the prize is behind Door 4 since Monty didn't open the door furthest to the right. It makes no difference whether you stay or switch doors. The authors in [2] also addressed this, and came to the same conclusion: "It really is a fifty-fifty decision in this case".

4.4.4 A Host who Mentally Flips a Coin

When presented with a choice of two goat-doors, Monty mentally flips a coin to choose which one to open. The contestant's chances of of winning are doubled if they switch doors. This is very similar to the classic version of the game.

4.4.5 A Game with Two Hosts

Let's say there are two hosts, M1 and M2. M1's strategy is to open the rightmost door and M2's strategy to mentally flip a coin when choosing between two goat-doors. The contestant randomly chooses a night where one of these two hosts were hosting, but has never seen M1 or M2 host before. While the chances of M1 or M2 being the host before the game are equiprobable, if we take into consideration that Door 3 has already been opened, then the probability of M1 being the host is $\frac{4}{7}$, since M1 prefers to open the rightmost door anyways. Thus, the probability of winning by switching is $\frac{4}{7}$.

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