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Solving the Rubik's Cube using Group Theory

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Abstract

While he was working in his mother's apartment in 1974, the professor of architecture from Budapest, Erno Rubik, had no idea he was inventing one of the most popular toys in history, the Rubik's Cube. As an estimated 350 million Rubik's cubes have been sold, and approximately one in every seven people have played with one (which is about 1 billion people) it is not surprising to see that the algorithm of solving the Rubik's cube has been applied to the field of mathematics. By using abstract algebra and more specifically, group theory, the Rubik's Cube, no matter what the starting configuration, can be solved. The notes on an intensive course written by Janet Chen guide this project by making the Rubik's cube a group where all of the possible moves are the elements in the group. By looking at the different subgroups and the moves in said subgroups, we can find the algorithm in which we can reconfigure the Rubik's cube back into its starting configuration. *A Mathematical Approach To Solving Rubik's Cube* by Raymond Tran also helped provide some different methods in achieving the correct configuration the Rubik's Cube. By utilizing these two texts we will prove that using these methods indeed gives us a viable solution to solve a Rubik's cube.

"If you are curious, you'll find the puzzles around you. If you are determined, you will solve them. " **-Erno Rubik**

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1 The Rubik's Cube and Subgroups

In order to talk about the Rubik's Cube as a group we must first define the notation we will be using. We will be using a notation developed by David Singmaster, where we refer to the faces as Right (r), Left (l), Up (u), Down (d), Front (f), and Back (b). When we are referring to the moves of a given face we will write them as uppercase letters of the corresponding face, R, L, U, D, F, and B. This will be in reference to one clockwise turn of the face when you are looking directly at it. When we twist the face twice in a clockwise direction we will refer to it as the face letter squared (eg. D^2). Lastly, when we twist the face once in a counterclockwise direction (or three times clockwise) we will refer to it as the inverse of the face (eg. D^{-1}). It is important to note that two moves will be considered the same if they result in the same end configuration. For example $\mathrm{D}^{-1} = \mathrm{D}^3$.

Definition 1.1. *Cubie and Cubicle:* A *cubie* is one of the 26 colored blocks on the Rubik's cube. A corner cubie has 3 visible faces while an edge cubie only has 2. When referring to the cubies we will name them based off of the starting location of the cubie, not on the colors of the faces. A *Cubicle*, on the other hand, is the space in which the cubie lives. If you rotate the face of the cube the cubicles do not move but the cubies do.

Remark. To name a corner or edge cubie we list the visible faces in clockwise order. For example the cubie that lies on the up, right, front corner of the cube is named **urf**. This cubie can also be referred to a **rfu** or **fur** if we do not care about the orientation of the cubie.

Definition 1.2. *Orientation:* The *orientation* of a cubie refers to the position the cubie has been twisted into no matter what cubicle the cubie is in. Thus the **urf** cubie can be in the **urf** cubicle but can be in the orientation of **fur** because the original Front face is on the Up face, the Up on the Right, and the Right on the Front. Thus, when we are referring to oriented cubies **rfu**, **fur**, and **urf** are not the same.

It is interesting to note the immense number of possible configurations of the Rubik's cube Since their are 8 corner cubies there are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8!$ possible positions of the corner cubies. Since their are 3 faces of a corner cubie, and therefore 3 orientations of a corner cubie, there are 3^8 possible ways the corner cubies could be oriented. In the same way we can look at the edge cubies and see their are 12! possible positions and with 2 different faces 2^{12} different orientations. Thus, if we combine all of the different possibilities we see that $2^{12}3^8 8!12!$ or 5.19×10^{20} potential configurations the Rubik's cube could be in- that is about 519 quintillion. However, it is important to note that some of these are actually not valid based on the starting configuration of the cube.

Theorem 1.1. The set of moves of the Rubik's Cube is a group denoted $(\mathbb{G}, *)$

Proof. In order to show that $(\mathbb{G}, *)$ is a group we must show that $\mathbb{G}$ closed under $*$, that a right identity exists, a right inverse exists, and $*$ is associative. Let M_1 and M_2 be two moves in $\mathbb{G}$. If M_1 and M_2 are moves then Let $M_1 * M_2$ is a move as well. Thus $\mathbb{G}$ is closed under $*$.

If we let e be the "empty move", which means it does not change the faces of the Rubik's cube at all, then $M * e = M$. Hence $\mathbb{G}$ has a right identity.

If M is an arbitrary move, then let the reverse steps of that move be M'. Then $M*M'$ means to first do all the moves of M and then undo all of the moves of M, leaving us in the same configuration we started in. Thus $M * M' = e$ and therefore M' is the inverse of M and every element of $\mathbb{G}$ has a right inverse.

If C is an oriented cubie, we will write $M(C)$ for the oriented cubicle that C ends up in after we apply the move M, with the faces of $M(C)$ written in the same order as the faces of C. That is, the first face of C should end up in the first face of $M(C)$, and so on. Let M_1 and M_2 be two moves in $\mathbb{G}$, then $M_1 * M_2$ is the move where we first do M_1 and then do M_2. The move M_1 moves C to the cubicle $M_1(C)$ and the move M_2 then moves it to $M_2(M_1(C))$. Thus, $(M_2 * M_1)(C) = M_2(M_1(C))$. To show $*$ is associative, we must show that $(M_1 * M_2) * M_3 = M_1 * (M_2 * M_3)$ for any moves M_1, M_2, and M_3. That is, we want to show that $[(M_1 * M_2) * M_3](C) = [M_1 * (M_2 * M_3)](C)$ for any cubie C. As previously stated in this proof we know that $[(M_1 * M_2) * M_3](C) = M_3([M_1 * M_2](C)) = M_3(M_2(M_1(C)))$. On the other hand, $[M_1 * (M_2 * M_3)](C) = (M_2 * M_3)(M_1(C)) = M_3(M_2(M_1(C)))$. So, $(M_1 * M_2) * M_3 = M_1 * (M_2 * M_3)$. Thus $*$ is associative.

Hence, $(\mathbb{G}, *)$ is a group. [1] □

Remark. From now on we will write the group operation $*$ as multiplication. For example instead of writing D $*$ R we will write DR, which means the move D followed by the move R.

Definition 1.3. *Commutator:* Let $M_1, M_2 \in \mathbb{G}$, the *commutator* is a subgroup of $\mathbb{G}$ such that $[M_1, M_2] = M_1 M_2 {M_1}^{-1} {M_2}^{-1}$

Theorem 1.2. $\mathbb{G}$ is not abelian.

Proof. Let M_1 be R and M_2 be B. Let us look at cubie **urf** in starting position. M_1M_2 moves **urf** to the **ulb** position. Now if we do moves M_2M_1 with **urf** in the starting position the **urf** cubie moves to the **bru** position. Since M_1M_2 and M_2M_1 produce different configurations, which we see from **urf** being in different positions, we can say that $\mathbb{G}$ is not abelian. □

Theorem 1.3. Let C_1 and C_2 be two different unoriented corner cubies, there is a move of the Rubik's cube which sends C_1 to C_1' and C_2 to C_2'.

Proof. Let C_1 and C_2 be two different unoriented corner cubies. Suppose that C_1 and C_1' share a face, without loss of generality call it f. Then F^n send C_1 to C_1'. Let C_2" denote the position of C_2 after F^n. If C_2" $= C_2'$ then we are done.

Suppose C_2" $\neq C_2'$. Since $C_1 \neq C_2$, we can't have $C_1' = C_2'$ or else two different cubies would occupy the same position. So C_1', C_2', and C_2" are three different corner cubies.

1. **Case 1** There exists a face, shared by C_2" and C_2' but not C_1'. Without loss of generality call it b, rotate it B^n times and we have $C_1 \to C_1'$ and $C_2 \to C_2'$.
2. **Case 2** C_2" and C_2' share a face with C_1'.
   (a) **Case a** More than 2 faces are shared with C_2", C_2', and C_1'. This is impossible based on the configuration of the Rubik's Cube.
   (b) **Case b** C_2" and C_2' do not share a face. Since every corner cubie shares a face with all but one other corner cubie, C_2" and C_2' each share at least one face with C_1'. C_2" and C_1' do not share at least one face, without loss of generality, call that face l. C_2" can be moved L^n times to share a face with C_2' but not share a face with C_1'. Without loss of generality call this face R, then C_2" can be moved to C_2' by R^n moves. Thus, $C_1 \to C_1'$ and $C_2 \to C_2'$.

Suppose that C_1 and C_1' do not share a face. Note, each corner cubie does not share a face with only one other corner cubie. Thus, C_1 can be moved by one move to any other adjacent corner and will now share a face with C_1', without loss of generality, call that face f. F^n moves will then put the C_1 cubie into the C_1' position. If C_2" $= C_2'$ then we are done.

Suppose C_2" $\neq C_2'$. Note, as mentioned earlier $C_1 = C_2$ and $C_1' \neq C_2'$. So C_1', C_2' and C_2' are three different cubies.

1. **Case 1** There exists a face shared by C_2" and C_2' but not C_1'. Without loss of generality call this face u. U^n can move C_2" and C_2' and therefore, $C_1 \to C_1'$ and $C_2 \to C_2'$.
2. **Case 2** There does not exist a face shared by C_2" and C_2' but not C_1'.
   (a) **Case a** Every face shared by C_2" and C_2' is shared with C_1'. This is impossible based on the configuration of the Rubik's Cube.
   (b) **Case b** C_2" and C_2' do not share a face. Since every corner cubie shares a face with all but 1 other cubie, C_2" and C_2' both share at least one face with C_1'. There is a face on C_2" that does not share a face with C_1' (otherwise they would be in the same position). Without loss of generality call that face d. Rotate C_2" D^n times such that C_2" and C_2' share a face (which is

possible since only the original position of C_2" will not share a face with C_2').

and C_2" does not share the same face that C_2' and C_1' share. Without loss of generality, call the face that C_2" and C_2', but not C_1, share r. Rotate the face R^n times until C_2" is in the C_2' position. Hence, $C_1 \to C_1'$ and $C_2 \to C_2'$.

Thus, there is a move of the Rubik's cube which sends C_1 to C_1' and C_2 to C_2'.

□

2 Cycle Notation for the Rubik's Cube

As to more easily describe what happens to the cubies when we apply different moves of the faces we use a modified cycle notation that describe the position, and later the orientations, of the Rubik's Cube cubies. Let us look at the move of the right face of the Rubik's cube. If you look at **rfu** after the move R is applied to it, it moves to the position **rub**. Meanwhile **rub** moves to **rbd**, **rbd** moves to **rdf** and **rdf** moves to **rfu**. We can write these moves, combined with what the move R does to the edge pieces, in cyclic notation as, R = (rfu, rub, rbd, rdf)(ru, rb, rd, rf).

Remark. We can look at all of the elements of $\mathbb{G}$ as products of disjoint cycles as follows:

D = (dlf, dfr, drb, dbl)(df, dr, db, dl)

R = (rfu, rub, rbd, rdf)(ru, rb, rd, rf)

U = (ulb, ubr, urf, ufl)(ul, ub, ur, uf)

L = (luf, lfd, ldb, lbu)(lu, lf, ld, lb)

F = (fur, frd, fdl, flu)(fu, fr, fd, fl)

B = (bul, bld, bdr, bru)(bu, bl, bd, br)

Using these cycles we can compose them to find the disjoint cycles of any other move.

One of the exercises in Janet Chen's course is to find the order of the move DR. This can be found by looking at the disjoint cyclic notation of the move DR. We can compose the moves D and R and we can visually see the cycles as we perform the moves on the Rubik's Cube. The move DR = (dlf, ruf, rub, rbd, dlb)(df, rf, ru, rb, rd, db, dl), note the cubies **drf**, **flu** and **ulb** stay in the same position so they are not part of the corner cycle moves. Since the order of the individual cycles are 5 and 7, the order of DR is 35. However, it is important to note that this is the order if you are only looking for the cubies to be in the same position, not the same orientation. If you want to find the order of this move such that you arrive at the same starting configuration with the same orientations then the order is 105. This is because each corner cubie has 3 faces and $3 \cdot 35 = 105$, this was also physically verified by applying the moves to the Rubik's cube.

One of the most useful exercises explored in Janet Chen's course is exploring the elements of the subgroup $H = \langle D^2, R^2 \rangle$:

Example 2.1. Let H be the subgroup of $\mathbb{G}$ generated by D^2 and R^2; that is, $H = \langle D^2, R^2 \rangle$. How many elements does H have?

First, let us look at D^2 and R^2.

D^2 = (dlf, drb)(dfr, dbl)(df, db) (dr, dl)

R^2 = (rfu, rbd)(rub, rdf)(ru, rd)(rb, rf)

By analyzing the composition of these two elements we can find all of the possible elements within H. Let us look at some of these compositions:

D^2R^2 = (dlf, rfu, drb)(df, db)(drf, dlb, rbu)(dr, dl, ru)(rf, rb)

R^2D^2 = (dlf, drb, rfu)(df, db)(drf, rbu, dlb)(dr, ru, dl)(rf, rb)

$D^2R^2D^2$ = (dlf, ruf)(ru, dl)(ru, rl)(rb, rf)(dlb, rub)

$R^2D^2R^2$ = (dlf, ruf)(df, db)(dl, ur)(rb, rf)(dlb, rub)

For brevity the rest of the cycles are not included but can be found by further composing D^2, R^2. Note, D^2 is never composed with D^2 and R^2 is never composed with R^2 because you would not change the configuration. After analyzing the compositions up to $(D^2R^2)^6$ and $(R^2D^2)^6$ the move that got us back starting configuration, the identity, was found. By comparing the different compositions the following element pairs were discovered. As stated before, if two moves give us the same configuration then we count them as the same element. Thus these pairs are the same element in H.

$D^2 = (R^2D^2)^5R^2$

$R^2 = (D^2R^2)^5D^2$

$D^2R^2 = (R^2D^2)^5$

$R^2D^2 = (D^2R^2)^5$

$D^2R^2D^2 = (R^2D^2)^4R^2$

$R^2D^2R^2 = (D^2R^2)^4D^2$

$(D^2R^2)^2 = (R^2D^2)^4$

$(R^2D^2)^2 = (D^2R^2)^4$

$(D^2R^2)^2D^2 = (R^2D^2)^3R^2$

$(R^2D^2)^2R^2 = (D^2R^2)^3D^2$

$(D^2R^2)^3 = (R^2D^2)^3$

$(D^2R^2)^6 = (R^2D^2)^6 = e$

Thus, after the element $(D^2R^2)^6 = (R^2D^2)^6 = e$, we would only get more elements in the group that equal elements we have already found. Hence, as seen above in the number of element pairs we have before reaching the identity, there are 12 elements in the group H. To make sure the group was closed and no elements were missed a group table of H was created (see next page).

This exploration of the group H let us see that the moves started becoming the inverse of each other, bringing the configuration back to the starting position (which can be seen when you look at the pattern of the pairs). This exploration will be useful later on when we are solving the Rubik's Cube we will be using the squares of moves composed with other moves, such as the commutator. This also helped identify patterns to look for within in the cyclic notation.

	R^2	D^2	$R^2 D^2$	$D^2 R^2$	$R^2 D^2 R^2$	$D^2 R^2 D^2$	$(R^2 D^2)^2$	$(D^2 R^2)^2$	$(R^2 D^2)^2R^2$	$(D^2 R^2)^2D^2$	$(R^2 D^2)^3$	e
R^2	e	$R^2 D^2$	D^2	$R^2 D^2 R^2$	$D^2 R^2$	$(R^2 D^2)^2$	$D^2 R^2 D^2$	$(R^2 D^2)^2R^2$	$(D^2 R^2)$	$(R^2 D^2)^3$	$(D^2 R^2)^2D^2$	R^2
D^2	$D^2 R^2$	e	$D^2 R^2 D^2$	R^2	$(D^2 R^2)^2$	$R^2 D^2$	$(D^2 R^2)^2D^2$	$R^2 D^2 R^2$	$(R^2 D^2)^3$	$(R^2 D^2)^2$	$(R^2 D^2)^2R^2$	D^2
$R^2 D^2$	$R^2 D^2 R^2$	R^2	$(R^2 D^2)^2$	e	$(R^2 D^2)^2R^2$	D^2	$(R^2 D^2)^3$	$D^2 R^2$	$(D^2 R^2)^2D^2$	$D^2 R^2 D^2$	$(D^2 R^2)^2$	$R^2 D^2$
$D^2 R^2$	D^2	$D^2 R^2 D^2$	e	$(D^2 R^2)^2$	R^2	$(D^2 R^2)^2D^2$	$R^2 D^2$	$(R^2 D^2)^2$	$R^2 D^2 R^2$	$(R^2 D^2)^2R^2$	$(R^2 D^2)^2$	$D^2 R^2$
$R^2 D^2 R^2$	$R^2 D^2$	$(R^2 D^2)^2$	R^2	$(R^2 D^2)^2R^2$	e	$(R^2 D^2)^3$	D^2	$(D^2 R^2)^2D^2$	$D^2 R^2$	$(D^2 R^2)^2$	$D^2 R^2 D^2$	$R^2 D^2 R^2$
$D^2 R^2 D^2$	$(D^2 R^2)^2$	$D^2 R^2$	$(D^2 R^2)^2D^2$	D^2	$(R^2 D^2)^3$	e	$(R^2 D^2)^2R^2$	R^2	$(R^2 D^2)^2$	$R^2 D^2$	$R^2 D^2 R^2$	$D^2 R^2 D^2$
$(R^2 D^2)^2$	$(R^2 D^2)^2R^2$	$R^2 D^2 R^2$	$(R^2 D^2)^3$	$R^2 D^2$	$(D^2 R^2)^2D^2$	R^2	$(D^2 R^2)^2$	e	$D^2 R^2 D^2$	D^2	$D^2 R^2$	$(R^2 D^2)^2$
$(D^2 R^2)^2$	$D^2 R^2 D^2$	$(D^2 R^2)^2D^2$	$D^2 R^2$	$(R^2 D^2)^3$	D^2	$(R^2 D^2)^2R^2$	e	$R^2 D^2 R^2$	R^2	$R^2 D^2 R^2$	$R^2 D^2$	$(D^2 R^2)^2$
$(R^2 D^2)^2R^2$	$(R^2 D^2)^2$	$(R^2 D^2)^3$	$R^2 D^2 R^2$	$(D^2 R^2)^2D^2$	$R^2 D^2$	$(D^2 R^2)^2$	R^2	$D^2 R^2 D^2$	e	$D^2 R^2$	D^2	$(R^2 D^2)^2R^2$
$(D^2 R^2)^2D^2$	$(R^2 D^2)^3$	$(D^2 R^2)^2$	$(R^2 D^2)^2R^2$	$D^2 R^2 D^2$	$(R^2 D^2)^3$	$D^2 R^2$	$R^2 D^2 R^2$	D^2	$R^2 D^2$	e	R^2	$(D^2 R^2)^2D^2$
$(R^2 D^2)^3$	$(D^2 R^2)^2D^2$	$(R^2 D^2)^2R^2$	$(D^2 R^2)^2$	$(R^2 D^2)^2$	$D^2 R^2 D^2$	$R^2 D^2 R^2$	$D^2 R^2$	$R^2 D^2$	D^2	R^2	e	$(R^2 D^2)^3$
e	R^2	D^2	$R^2 D^2$	$D^2 R^2$	$R^2 D^2 R^2$	$D^2 R^2 D^2$	$(R^2 D^2)^2$	$(D^2 R^2)^2$	$(R^2 D^2)^2R^2$	$(D^2 R^2)^2D^2$	$(R^2 D^2)^3$	e

3 Analyzing Position and Orientation

Thus far as we have evaluated what the different moves, or elements, of the Rubik's Cube we have not really focused on the orientation of the cubies, just where one move sends the cubie. We will be using the same notation as Janet Chen in order to track the orientation, as well as the position, of the corner and edge cubies. We have already discussed how to write the positions of the cubies in cyclic notation, while not worrying about orientation, but now we will look at the position of the cubies in a slightly different way. We are going to label each edge and corner cubie with a number as follows:

For the corner cubies

1 on the u face of the **ufl** cubie

2 on the u face of the **urf** cubie

3 on the u face of the **ubr** cubie

4 on the u face of the **ulb** cubie

5 on the d face of the **dbl** cubie

6 on the d face of the **dlf** cubie

7 on the d face of the **dfr** cubie

8 on the d face of the **drb** cubie

For the edge cubies

1 on the u face of the **ub** cubie

2 on the u face of the **ur** cubie

3 on the u face of the **uf** cubie

4 on the u face of the **ul** cubie

5 on the b face of the **lb** cubie

6 on the b face of the **rb** cubie

7 on the f face of the **rf** cubie

8 on the f face of the **lf** cubie

9 on the d face of the **db** cubie

10 on the d face of the **dr** cubie

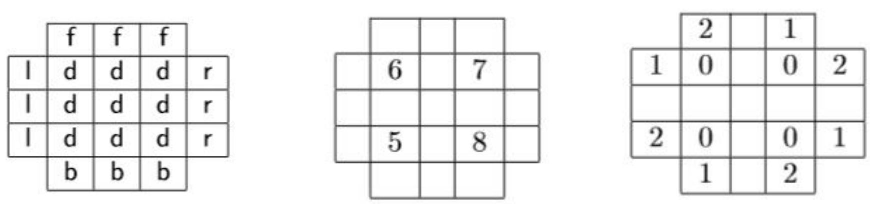
11 on the d face of the **df** cubie

12 on the d face of the **dl** cubie

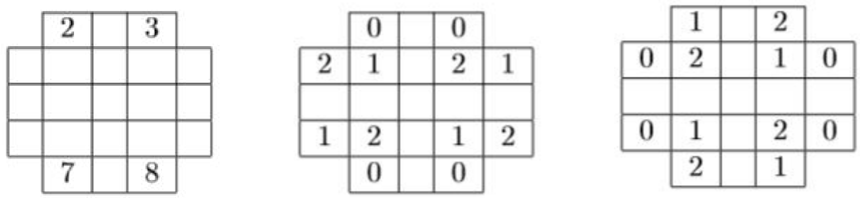
For the corner cubies we can write this as an element σ of S_8 (the element of S_8 which moves the corner cubies from their starting positions to the new positions) where $\sigma : \{1,2,3,4,5,6,7,8\} \to \{1,2,3,4,5,6,7,8\}$. Thus σ is defined by the placement of the corner cubies in their resulting cubicles after a specific move has been performed. For the edge cubies we can write this as an element of τ of S_{12} where $\tau : \{1,2,3,4,5,6,7,8,9,10,11,12\} \to \{1,2,3,4,5,6,7,8,9,10,11,12\}$. [1] Thus τ is defined by the placement of the edge cubies in their resulting cubicles after a specific move has been performed. In Janet Chen's notes she uses the letter names of the cubies to talk about σ and τ, however for our case we are going to just use the position numbers coupled with the orientations to analyze where the cubies are being sent.

For example, the move R would change the starting position of σ = (1, 2, 3, 4, 5, 6, 7, 8) to σ = (1, 7, 2, 4, 5, 6, 8, 3). Here we see the 2 cubie moves to the 3 position, the 3 to the 8, the 8 to the 7, and the 7 to the 2 position.

Now that we have labeled all of the edge and corner cubies we can now look at the corresponding orientations. Let us first look at the corner cubies. Note that there are three different faces of the corner cubie, thus the cubie can be in three different orientations. Additionally, each corner cubie has one numbered face to identify the position, mark this numbered face 0. From there, going clockwise around the faces of the cubie label the subsequent face 1, and the last face 2. In order to get a better understanding of what that looks like let us take a look at a visual representation provided by Janet Chen [1]. In the first image we see the letter of the face on the flattened downside of the Rubik's Cube, then we see how those cubes are labeled for their position, and lastly, we see the orientation associated with those positions.



In order to look at the orientations of all of the corner cubies at the same time we can write the resulting orientation of all of the corner cubies after a move has been performed as an 8-tuple, x. Again using Janet Chen's notation, for any i between 1 and 8, find the cubicle face labeled i; let x_i be the number of the cubie face living in this cubicle face. [1] Therefore, $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$, were the sub numbers are in reference to the position of x, or σ. Thus x_1 is the orientation of the cubie in the 1 labeled position. Note, x in the starting position is, $x = (0, 0, 0, 0, 0, 0, 0, 0, 0)$. Each x_i is equivalent to counting the number of clockwise twists the cubie i is away from having its 0 face in the numbered face of the cubicle. Let us take a look at how the move R affects the orientations of the corner cubies of the Rubik's Cube. Again, we will be using images provided by Janet Chen [1].



In the first two images we see the starting position and orientation, respectively. In the last image we see the orientation after the move R is applied to the Rubik's Cube. In the cubies that are unaffected, and not shown, their orientation stays at 0. However the orientations of x_2, x_3, x_7, and x_8 do change. As seen in the above image, $x_2 = 1$, $x_3 = 2$, $x_7 = 2$ and $x_8 = 1$. We can write the value x after R as $x = (0, 1, 2, 0, 0, 0, 2, 1)$

This same process can be used for the edge cubies. We can label each face with a positional number on it 0 and the other face 1. Because there are 12 edge cubies, we can write the resulting orientation of the edge cubies after a move as an 12-tuple, y. Such that $y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$, where, like before with x, y_i is the number in the edge cubicle face i, where i is the position we get by looking at τ. Thus y_1 is the orientation of the cubie in the 1 labeled edge position. Observe that in the starting position $y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. Now lets take a look at how R affects the orientations of the edge cubies. R only affects the cubies y_2, y_6, y_7, and y_{10}. Interestingly, when R is rotated the orientation stays exactly the same and $y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. In fact when we studied x and y after every move we found only the moves F and B affected the orientation of the edge cubies, as seen below.

Example 3.1. Given the Rubik's Cube is in the starting position, analyze x and y for the single moves of the Rubik's Cube: D, U, L, R, F, B.

1. **D**

   $x = (0, 0, 0, 0, 0, 0, 0, 0)$

   $y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

2. **U**

   $x = (0, 0, 0, 0, 0, 0, 0, 0)$

   $y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

3. **L**

   $x = (2, 0, 0, 1, 2, 1, 0, 0)$

   $y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

4. **R**

   $x = (0, 1, 2, 0, 0, 0, 2, 1)$

   $y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

5. **F**

   $x = (1, 2, 0, 0, 0, 2, 1, 0)$

   $y = (0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0)$

6. **B**

   $x = (0, 0, 1, 2, 1, 0, 0, 2)$

   $y = (1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0)$

These 8-tuples and 12-tuples will be made of use later on as we solve the Cube to help us prove that we get the correct orientations and positions.

One of the most important moves utilized when solving the Rubik's Cube is the commutator. This next example displays not just how the position of the cubies change when the commutator [D,R] is used but also keeps track of the orientations, which is necessary to take into account when we actually begin to solve the cube.

Example 3.2. Write the commutator [D,R] in disjoint cycle notation (be careful to keep track of the orientations of the cubies). What is the order of [D,R]?

[D,R] = DRD^{-1}R^{-1} = (dlf, dfr, lfd, frd, fdl, rdf)(df, dr, br)(drb, bru, bdr, ubr, rbd, rub)

$\sigma = (1, 2, 8, 4, 5, 7, 6, 8)$

$\tau = (1, 2, 3, 4, 5, 10, 7, 8, 9, 11, 6, 12)$

$x = (0, 0, 2, 0, 0, 2, 0, 2)$

$y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

Looking at the disjoint cycles of [D,R] we see that the individual orders are 6, 3, and 6. Thus, given that the lease common multiple is 6, the order of [D,R] is 6.

Note, from now on we will mostly be using the notation of σ, τ, x, and y when discussing the movement of the cubies based on different elements of $\mathbb{G}$ in order to simplify notation. The table of σ, τ, x, and y for [D,R] is shown in Table 1 in Section 4.

4 Solving the Rubik's Cube

After compiling the necessary notation, definitions, theorems, and practice it is finally time to begin solving the Rubik's Cube. Considering the Rubik's Cube has 43 quintillion possible valid positions *A Mathematical Approach To Solving Rubik's Cube* was referenced to provide guidance on where to start (as well as where to go and how to end). In this text there are two methods to solving the Rubik's Cube, we will be using the second, "Layer-by-Layer" Method. In this method the top layer, or the Up face, is solved first, then the middle layer edge pieces of the Front, Right, Back, and Right faces, and finally the Down face.

Before talking about the actual moves we will be using let us take a look at how we will be analyzing each move to make sure the cubies we are moving get into the cubicles we want them to be in. In Table 1 on the next page we are looking at the commutator move [D, R]. Every move table has four parts: σ, τ, x, and y. As we talked about before, σ and τ refer to the positions of the corner and edge cubies, respectively. Where x and y refer to the orientations of the edge cubies. Note that sometimes there will be negative numbers or positive numbers that are not 0,1, or 2 in the orientation columns thus, every corner cubie orientation should be taken Mod 3 and every edge cubie orientation should be taken Mod 2. This table was constructed using Macros formulas in Excel such that when we inputed the move, such as D, it would output the position and orientation in the column for that move of every cubie. A couple of the following moves were found done out by hand but that was found to be a very long and tedious processes. Therefore, this program was constructed in order to more easily analyze moves that have many face twists in them. As said before, the table is broken up into four parts where each move starts with the cubies in their starting position thus, $\sigma = (1, 2, 3, 4, 5, 6, 7, 8)$, $x = (0, 0, 0, 0, 0, 0, 0, 0)$, $\tau = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$, and $y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. The cubies resulting positions and orientations by the move described in the table are given under the column of the last twist in that move, for example the resulting positions and orientations for the table below are $\sigma = (1, 2, 8, 4, 5, 7, 6, 3)$, $x = (0, 0, -1, 0, 0, 2, 0, -1)$, $\tau = (1, 2, 3, 4, 5, 10, 7, 8, 9, 11, 6, 12)$, and $y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. As we talk about the moves used to solve the Rubik's Cube we will reference the tables provided in the Appendix in order to prove that the cubies do in fact end up where we need them to go. Now that we have all of the tools needed to solve the Rubik's Cube let us begin to talk about the actual moves used to solve the Cube.

Table 1: $DRD^{-1}R^{-1}$ Commutator

σ	D	R	D^{-1}	R^{-1}		x	D	R	D^{-1}	R^{-1}
1	1	1	1	1		0	0	0	0	0
2	2	6	6	2		0	0	1	1	0
3	3	2	2	8		0	0	2	2	-1
4	4	4	4	4		0	0	0	0	0
5	8	8	5	5		0	0	0	0	0
6	5	5	7	7		0	0	0	2	2
7	6	7	3	6		0	0	2	1	0
8	7	3	8	3		0	0	1	0	-1
τ						y				
1	1	1	1	1		0	0	0	0	0
2	2	7	7	2		0	0	0	0	0
3	3	3	3	3		0	0	0	0	0
4	4	4	4	4		0	0	0	0	0
5	5	5	5	5		0	0	0	0	0
6	6	2	2	10		0	0	0	0	0
7	7	11	11	7		0	0	0	0	0
8	8	8	8	8		0	0	0	0	0
9	10	10	9	9		0	0	0	0	0
10	11	6	10	11		0	0	0	0	0
11	12	12	6	6		0	0	0	0	0
12	9	9	12	12		0	0	0	0	0

4.1 Stage 1- Top Layer

The goal for solving this layer is to first get a cross of all matching colors on the Up face, making sure that the edge pieces of the cross align with the colors of the middle pieces on the Back, Left, Front, and Right faces. After the cross is formed, the corner pieces that match the Up face color and the two other side face colors (Front, Right, Back, and Left) need to be put into the right position, as well as orientated correctly.

In order to begin to make the cross we first want to start by trying to place one of the cubies that belong on the Up face into the correct cubicle position. We are going to try to place the correct cubie into the **ub** cubicle, which is cubicle position 1 from our definition of τ. For example if your Up face is green and your Back face is red, we are looking for the green and red edge cubie to move into the **ub** cubicle. Right now do not worry about orientation, we are just trying to get the edge cubie in the right position. Since there are 12 edge cubicles, there are twelve possible positions the desired edge cubie could start in. If the cubie is already in cubicle 1, **ub**, then we do not need to make any moves. If the cubie is in cubicle 2, **ur** then we can do the move U^{-1} (which is the same thing as U^3) and the cubie in **ur** will now be in the **ub** cubicle. This can be verified by composing the move U, see section 2, three times. Listed below are all of the moves that can be performed depending on which cubicle the cubie we need to move starts in. The list states the cubicle position τ, the name of the cubicle, and then the move needed to get the cubie in that cubicle to **ub**. All of these moves can be verified by hand, doing the moves on the Rubik's Cube, or by composing the moves listed in the beginning of section 2.

3- uf: U^2 **4- ul**: U **5- lb**: B^{-1}

6- rb: B **7- rf**: RU^{-1} **8- lf**: $\mathrm{L}^{-1}\mathrm{U}$

9- db: B^2 **10- dr**: DB^2 **11- df**: $\mathrm{D}^2\mathrm{B}^2$

12- dl: $\mathrm{D}^{-1}\mathrm{B}^2$

After performing one of these moves you will have the edge cubie in the right position of **ub**. If the cubie is oriented correctly such that the correct color matches up with the Up and Back faces then this step can be skipped. However, if the cubie is unoriented then we must perform one more move to get it in the correct orientation. The move we can use to change the orientation is $\mathrm{B}^{-1}\mathrm{R}^{-1}\mathrm{U}^{-1}$. B^{-1} takes the cubie in **ub** and moves it to **rb**, which is y_6. If we look at the move B in section 3, we see that it changes the orientation of y_6, and if we look at the composition of B three times (since $\mathrm{B}^{-1} = \mathrm{B}^3$) we will see that B^{-1} also changes the orientation. Now that we have changed the orientation we can perform R^{-1}, which does not change the orientation but sends the cubie to **ur**, and then U^{-1}, which sends the cubie back to **ub**, now in the correct orientation.

One edge cubie should now be lined up with the Up face middle cubie, as well as next to the correct color on the middle cubie of the Back face. Now turn the whole Rubik's Cube such that the cubie we just placed is now the **ul** cubicle and the cubicle we are trying to solve next is the **ub** cubicle. Do not worry about orientation, we are just trying to get the cubie into the correct position. Like we did for the first cubicle we are going to go through all of the possible positions the edge cubie could be in and then provide the move for that position to get the cubie into the **ub** cubicle. There are 11 possible positions the edge cubie could be in, as it cannot be in the **ul** cubicle, position 4, because we already solved that cubicle. If the correct cubie is in position 1, **ub** then it can stay there, if not do the one of the following moves based off of what cubicle the cubie you want to move to **ub** is in. Note these moves can be verified by hand, doing the moves on the Rubik's Cube, or by composing the moves listed in the beginning of section 2.

2- uf: RB **3- ul**: FR^2B **5- lb**: B^{-1}

6- rb: B **7- rf**: R^2B **8- lf**: $F^{-1}D^2B^2$

9- db: B^2 **10- dr**: DB^2 **11- df**: D^2B^2

12- dl: $D^{-1}B^2$

Note none of these moves involve the Up or Left face, thus it will not disturb the correctly placed edge cubie in the **ul** cubicle. If the cubie in the **ub** cubicle is already correctly oriented then we can begin positioning the third Up face edge cubie, if not we must first orient the edge cubie we just positioned into the **ub** cubicle. The move we will use to change the orientation is $B^2D^{-1}R^{-1}B$. Again, this move does not involve the Up or Left face so it will keep the already placed edge cubie in the **ul** cubicle in its place. The move starts with B^2, which keeps the orientation the same and moves the cubie in the **ub** cubicle to the **db** cubicle. Then the move D^{-1} moves the cubie to the **dr** cubicle, followed by the move R^{-1} which moves the cubie to the **rb** cubicle. Neither of those two twists affect the cubie's orientation. Lastly is a move of B, which moves the cubie from **rb** to the **ub** cubicle and changes the orientation, as seen in Section 3. We now have a correctly oriented cubie in the correct position of **ub**.

Next is the third edge cubie on the Up face. We need to make sure not to disturb the other two already oriented cubies. Hold the Rubik's Cube such that the positioned edge pieces are in the **ul** and **uf** cubicles. We are going to be solving the **ub** cubicle. The edge cubies in the **ul** and **uf** cubicles are already solved so the cubie that belongs in **ub** cannot be in those positions and if the cubie is already in position 1, **ub**, then it can stay in the cubicle. If the cubie that belongs in the **ub** cubicle is anywhere besides the **lf** cubicle, position 8, then the following listed steps can be taken. Note of these moves can be verified by hand, doing the moves on the Rubik's Cube, or by composing the moves listed in the beginning of section 2.

2- uf: RB **5- lb**: B^{-1} **6- rb**: B

7- rf: R^2B **9- db**: B^2 **10- dr**:DB^2

11- df: D^2B^2 **12- dl**: $D^{-1}B^2$

If the edge cubie is in the **lf** cubicle we must move a positioned edge cubie out of its cubicle and then replace it because there is no way to move the cubie that needs to go to **ub** without disturbing a correctly placed Up face edge cubie. To do this we are going to perform the move $F^{-1}D^2B^2F$. This move twists the Front face once counterclockwise so that the **lf** cubie moves to the **fd** cubicle. Then rotate the Down face twice so that it brings the cubie to the **db** cubicle. Next rotate the Back face twice which moves the cubie from **bd** to the **ub** cubicle, the desired position. Note none of the previous moves affected the correctly positioned **ul** cubie but it did move the correctly positioned **uf** cubie into the **fl** cubicle. To fix this, we do the last move of a Front face counterclockwise twist which puts the cubie correctly positioned in the **ub** cubicle again. Also notice that the counterclockwise twist does not affect the **ul** or the **ub** cubicles.

Now that we have the edge cubie in the right position of **ub** with correctly positioned edge cubies in the **ul** and **uf** cubicles, we can use the Back, Down, and Right faces to correctly orient the third edge cubie, since moving those faces will not affect the correctly oriented and positioned cubies in the **ul** and **uf** cubicles. We can use the same move we used to orient the second edge cubie, $B^2D^{-1}R^{-1}B$, because it does not involve the Up, Left, and Front faces. To see the justification and the path the cubie takes, refer to the paragraph earlier about fixing the orientation of the second edge cubie.

Placing the fourth edge cubie is a little more difficult because we have to be sure not to displace any of the other three correctly placed edge cubies, and if we do we need to replace them back into the same cubicle. Hold the Rubik's Cube such that the face without the correct cubie in the cubicle is the Front face, such that the cubicle we are trying to get the cubie in is **uf**. If the desired cubie is already oriented on the Front or Down face then the move is very simple. If the cubie is oriented correctly on the Down face we need to simply rotate the Down face once, twice, or once counter clockwise until it is on the front face in the **fd** cubicle. If the edge cubie is in position 9, **db**, rotate the Down face twice, if the cubie is in position 10, **dr**, rotate the Down face once counterclockwise, if the edge cubie is in position 12, **dl**, rotate the Down face once, and lastly if the edge cubie is in position 11, **df**, it can stay there. Note that a Down face twist does not affect any of the properly placed edge cubies on the Up face. With the correct cubie in the **fd** cubicle in the right orientation we can twist the Front face twice such that the cubie ends up in the **uf** cubicle. The only edge cubie that the front twist move affects is the **uf** cubie, thus the rest of the properly placed cubicles will remain in place. If the edge cubie is on the Front face, properly oriented, then the moves are as follows: if the edge cubie is in position 7, **rf**, rotate the Front face once counterclockwise, if the cubie is in position 8, **lf**, rotate the Front face once, if the edge cubie is in position 11, **df**, rotate the Front face twice, and lastly if the edge cubie is in position 3, **uf**, it can stay there.

If the properly oriented edge cubie is on one of the edge cubicles not on the Down or the Front face (note- the edge cubie cannot be in any of the other Up face cubicles besides the **uf** cubicle because then it would be one of the already properly placed edge cubies), then we have one extra step. If the edge cubie is in the **rb** or **lb** cubicle, position 6 and 7, then we must get the cubie onto the Down face. Once we get the cubie into the down face we can follow the steps mentioned in the previous paragraph about what to do what the cubie is on the Down face. In order to get this edge cubie onto the Down face we are going to have to displace the properly placed edge cubie in the **ub** cubicle, and then reposition it again. If the edge cubie we want to move is in the **rb** position then twist the Back face once counterclockwise, moving the desired cubicle to the **db** cubicle, then rotate the Down face twice, which will move the cubie we need to get to the **fd** cubicle. During those two moves the cubie in the **ub** cubicle was moved to the **rb** cubicle during the Back counterclockwise twist and then kept there during the Down face twist. Thus the final move is a clockwise twist of the Back face and we have the cubie correctly positioned in the **ub** cubicle again. The process is exactly the same for if the cubie we are moving to **uf** position is in the **lb** cubicle with the exception that we start with one twist of the Back face and end with one counterclockwise twist of the Back face. Now you will have your cubie in the **fd** cubicle and it can be moved to the **uf** cubicle by two twists of the Front face. If the fourth edge cubie is in the wrong orientation then we have a couple of extra moves we must execute.

If the last edge cubie we need to move to the **uf** cubicle is oriented incorrectly then our first goal is to get the cubie into one of the Front face cubicles. If the cubie is already on the Front face we can skip this step. If the edge cubie is on the Down face then we can move the edge cubie to the Front face into the **df** cubicle. In the same way we would move the cubie if it were oriented correctly, when the edge cubie is in position 9, **db**, rotate the Down face twice, when the cubie is in position 10, **dr**, rotate the Down face once counterclockwise, when the edge cubie is in position 12, **dl**, rotate the Down face once, and lastly when the edge cubie is in position 11, **df**, it can stay there. Note that a twist of the Down face does not disturb any of the properly placed edge cubies on the Up face. However, if the edge cubie is on one of the edge cubicles not on the Down or the Front face, then we have to do the same step described in the previous paragraph. If the edge cubie is in the **rb** or **lb** cubicle then we must get the cubie onto the Down face so we can follow the steps mentioned in the beginning of this paragraph. As mentioned before, in order to get this edge cubie onto the Down face we are going to have to displace the properly placed edge cubie in the **ub** cubicle, and then reposition it again. If the edge cubie we want to move is in the **rb** position then twist the Back face once counterclockwise, moving the desired cubicle to the **db** cubicle, then rotate the Down face twice, which will move the cubie we need to the **fd** cubicle. During those two moves the cubie in the **ub** cubicle was moved to the **rb** cubicle during the Back counterclockwise twist and then kept there during the Down face twist. Thus the final move is a clockwise twist of the Back face and we have the cubie correctly positioned in the **ub** cubicle again. As said previously, the process is exactly the same for if the cubie we are moving to **uf** position is in the **lb** cubicle with the exception that we start with a twist of the Back face and end with a counterclockwise twist of the Back face. Now that we have discussed how to get the unoriented cubie into a Front face edge cubicle we can now talk about how to orient that cubie and move it to the **uf** cubicle. With the desired cubie unoriented on the Front face it will be in one of three positions, the **uf** cubicle, the **fd** cubicle, or the **fl/fr** cubicles, and they all have different moves that will position them in the correct orientation in the **uf** cubicle.

1. *Edge Cubie unoriented in the* **uf** *cubicle*

   With the unoriented cubie in the **uf** cubicle we can perform the move $\mathrm{FU^{-1}RU}$ to get the cubie with the correct orientation back into the **uf** cubicle. In Table 2 in the Appendix it is shown that none of the other Up face edge cubies are affected by this move in their position or orientation (they are y_1, y_2, and y_4). Also we see that it changes the orientation of **uf** (y_3) and puts it back into the **uf** cubicle. Table 2 also shows where each cubie moves to during each twist of the move.

2. *Edge Cubie unoriented in the* **fd** *cubicle*

   If the unoriented edge cubie is in the **fd** cubicle we can perform the move $\mathrm{F^{-1}R^{-1}D^{-1}RF^2}$ to get the cubie into the **uf** cubicle. Looking at Table 3 we see that none of the other Up face edge cubies are affected by this move in their position or orientation. Also we see that it changes the orientation of **fd** (y_{11}) and moves it to the **uf** cubicle.

3. *Edge Cubie unoriented in the* **fl** *or* **fr** *cubicle*

   If the cubie is in the **fl** cubicle rotate the Front face twice such that the cubie is in the fr spot. This does not affect any of the other three edge cubies because none of them share a face with the Front Face. Also, the rotation of the front face twice does not affect the orientation of the cubie now in the **fr** cubicle. With the unoriented cubie in the **fr** cubicle we can perform the move $\mathrm{R^{-1}D^{-1}RF^2}$ to get the cubie with the correct orientation in the **uf** cubicle. If we take a look at Table 4 we see that none of the other Up face edge cubies are affected by this move. Additionally, this move changes the orientation of **fr** (y_7) and puts it in the **uf** cubicle (y_3).

After completing the moves for either the oriented, or unoriented fourth edge cubie we have completed the cross for the edge cubies on the Up face and we can now begin to get the corners oriented and positioned on the Up face.

With the edge cubies forming a cross on the Up face we can look to place the correct corner cubies in the Up face cubicles. We will be using the move $DRD^{-1}R^{-1}$, or [D,R]. It is interesting to note that commutators are used in conjunction with other face twists as we find more and more moves to continue to solve the Rubik's Cube. Let's take a look at what this move does in Table 1 (see the beginning of Section 4). When we use this commutator it keeps three of the top corner cubies in their place, in the same orientation, and switches 3 and 8, which is **dbr** and **ubr**. Note that it does not change the orientation of any of the edge cubies on the Up face so the edge cubies that we just properly positioned will stay in place. Now that we know the move that we will use and what corner faces they will switch, **dbr** and **ubr**, we need to look at the two possible configurations we will come across and how to rearrange them.

The first possible configuration we will have is if the corner cubie we need to position is on the Down face. If this is the case, hold the cubie such that the cubicle we need to move the cubie into is in the **ubr** cubicle, position 3 by the definition of σ, then twist the Down face either once, twice, once counter clockwise, or don't twist it at all, such that we end up with the cubie we are moving in the **dbr** cubicle. If the cubie we need to move is in position 5, **dbl**, then twist the Down face once counterclockwise, if the cubie is in position 6, **dlf**, then twist the Down face twice, if the cubie is in position 8, **dfr**, then twist the Down face once clockwise, and if the cubie is in position 8, **dbr**, then keep the cubie where it is. Then depending on the orientation of the cubie we will need to perform [D, R] either once, twice, or five times. This is because if we are trying to switch move the cubie from **dbr** and move it to the **ubr** cubicle, the first time you perform the move it will switch these two cubies and change their cubicle position. If the move is done a second time then the two cubies will switch again moving the cubie that started in the **dbr** cubicle to be placed back in the **dbr** cubicle. Thus if the move is done a third time the cubie in **dbr** will then be in the **ubr** cubicle. The same arguement can be made to show this is why the **dbr** cubie ends up in the correct cubicle of **ubr** on the fifth iteration of the move [D,R]. Note there are three different orientations of the cubie and the move [D, R] switches x_8, **dbr**, and x_3, **ubr**, every time, changing the orientation by two twists. If you refer back to Table 1, it is verified that x_8 and x_3 switch positions and change by an orientation of -1, or two twist. Thus, depending on the orientation we require the correct orientation will occur during either the first, second, or third time the cubie gets into the correct cubicle. And the cubie is moved into the correct cubicle during the first, third, and fifth iterations of the move [D, R].

The second possible configuration we will have is if the corner cubie we need to position is already on the Up face, either in or not in the right position. What we want to do is get that cubie onto the Down face and then make the same move we just discussed when the cubie starts on the Down face. With the cubie on the Up face, in position 1-4, or cubicles **ufl, urf, ubr** or **ulb**, hold the Rubik's Cube such that the cubie we need to move is in the **ubr** cubicle. Then use the [D, R] commutator to move that cubie to the **dbr** cubicle. Then follow the move instructions for when the cubie we need is on the Down face, as addressed above.

Repeat this process for each of the four corner cubies we need to place in the Up face of the Rubik's cube. As noted before, repeating this process for each corner cubie will not affect the rest of the Up face cubies because the only Up face cubie that is affected is **ubr**, which is the one you are working with to switch with **dbr**. Once all of the Up face cubies are placed we are done with the first layer and we can move onto solving the middle layer edge cubies on the Front, Right, Back, and Left faces.

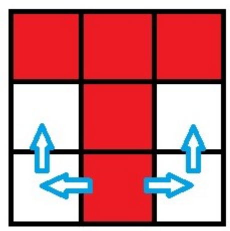
4.2 Stage 2- Middle Layer

The goal for solving the middle layer is to get all of the middle layer edge pieces of the Front, Left, Back, and Right faces into the right position with the correct orientation. Some of the edge pieces might already be in the correct position and orientation and this works to our advantage because it means less moves in order to complete the Cube. We will see that the moves we preform in this layer will only affect the middle edge cubie we want to change- not any other edge cubies in that middle layer. Thus, we do not have to worry about possibly switching around already correctly placed middle layer edge cubies. It is also important to note that none of these moves will affect the top layer (if done correctly). As we work down the layers of the Rubik's Cube more cubies are moved to the correct cubicles, which gives us longer, more complicated moves since we have less cubies that we can disturb without consequence.

There are two possible positions for where the edge cubie needed in the middle layer could start: on the Down face or in the wrong cubicle or orientation in the middle layer. If it is in the Down face we need only move the edge cubie from the Down face to the correct cubicle in the middle layer. If it is in the middle layer already, in the incorrect cubicle, we must first get the cubie to the Down face and then preform a move to get it back to the correct cubicle in the middle layer.

1. *Edge Cubies on the Down Face*

   If you have an edge cubie on the Down face, one of the edge cubie faces will be on the Down face and the other will be on either the Front, Right, Back, or Left face. The first step is to get the color that is on the Front, Right, Back or Left face directly under the corresponding matching middle cubie color, as seen in the figure below, where the Down edge cubie matches the same color as the middle cubie on that face. The color on the Down face of this edge cubie must match the color of the face to the left or right of the face the edge cubie is on. At this point the cubie needs to go to the cubicle on the left of the middle cubie or to the cubicle right of the middle cubie, as seen in the figure below. For instance if we are looking at the Front face, the edge cubie in the **fd** cubicle must either move to the **fr** cubicle or the **fl** cubicle, depending on the color on the Down face. If the color on the Down face of the cubie matches the face to the right then the edge cubie must go to the right, if not the cubie must go to the left. When performing the following moves we are going to hold the Rubik's cube such that we are always trying to move the desired cubie to is the **rf** edge cubicle. Thus, hold the Rubik's cube so the matched up face (shown below) is the Front face if you are moving the edge cubie to the right, which moves the edge cubie we are working with from the **fd** cubicle to the **fr** cubicle. If the edge cubie needs to be moved to the left, as shown in the figure, hold the Rubik's Cube such that the face shown below is the Right face where the edge cubie needs to be moved from the **rd** cubicle to the **fr** cubicle.



If the cubie we are moving to must go to the right middle layer edge cubicle of the face it is on, the cubie should be starting in the **df** cubicle. In order to get the cubie from **fd** to **fr** we must perform the move $D^{-1}R^{-1}DRDFD^{-1}F^{-1}$, which equals $[D^{-1}, R^{-1}][DF]$. Since this move switches the positions of **fd** and **fr**, that is why we needed to rotate our whole Rubik's Cube such that the cubie we are moving is in the **fd** cubicle. We can see this in Table 5 where the only two edge cubies this affects are in the positions 7 and 12, which are **rf** and **df**, the two we are looking to switch. Also, note that it keeps the top layer corner cubies in the same position and orientation as well so the first layer remains intact.

If the cubie must go to the left middle layer edge cubicle of the face it is on, the cubie should be starting in the **dr** cubicle. Then perform the move $DFD^{-1}F^{-1}$ $D^{-1}R^{-1}DR$, which equals $[DF]$ $[D^{-1}, R^{-1}]$. This move switches the positions of **rd** and **fr**, which is why we to rotate our whole Rubik's Cube such that the cubie we are moving is in the **rd** cubicle. In Table 6, note that the only two edge cubies this affects are in the positions 9 and 12, which are **rf** and **dr**, the two we need to switch. Again note that it keeps the Up Face corner cubies in the same position and orientation as well so the first layer remains intact.

2. *Edge Cubies in the Middle Layer*

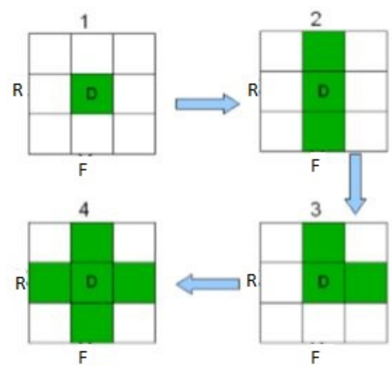
   If the cubie is already in the middle layer but in the wrong position then either move from above can be used to get the edge cubie onto the bottom layer. Make sure the Rubik's Cube is held such that the cubie that desired to be moved to the bottom layer is in the **rf** cubie, then either $\mathrm{DFD^{-1}F^{-1}}$ $\mathrm{D^{-1}R^{-1}DR}$ or $\mathrm{D^{-1}R^{-1}DRDFD^{-1}F^{-1}}$ can be used. These two moves are viable options because, as mentioned before, the moves switch the cubie that is in the **rf** cubicle to either the **df** or the **dr** cubicle, depending on which move chosen. After the cubie is moved to the bottom layer, step 1 can be used to put the edge cubie into its proper place

This process should be repeated for each of the four cubies that need to be moved into the edge cubicles. After each of the middle edge cubies are placed we can solve the bottom layer of the Rubik's Cube.

4.3 Stage 3- Bottom Layer

We have finally reached the bottom layer. Fortunately, we are almost at our end goal but unfortunately, that means are moves are more complicated and much longer in order to not disturb all of the properly positioned and oriented cubies in the top two layers. While turning the faces in this layer it is important to be precise and keep track of all of the moves we are making. Some of these moves are very long sequences and it is very easy to skip one or turn a face in the wrong direction.

In the same fashion as the top layer we are going to begin the bottom layer by getting the edge cubies into a cross. After finishing the middle layer your Down face will be in one of four positions: No edge cubies in the right orientation, one edge cubie in the right orientation, two edge cubies in the right orientation on opposite faces, two edge cubies in the right orientation on adjacent faces, or all four cubies in the right orientation. The figure below, provided by Raymond Trans' article [2], shows these orientations and how they fall in respect to the Right and Front faces (this is because we will be using the right and Front faces for the subsequent move). If all four edge cubies already have the right position then we can move on to the next step, if not we must perform a few moves first. If the cube is in one of the other three configurations we will use the move $\mathrm{RDFD^{-1}F^{-1}R^{-1}}$ which is $\mathrm{R[DF]R^{-1}}$. This move rotates the Down face through the four configurations mentioned above. Note in Table 7 we see that the top 4 corner cubies are kept in the same position and orientation (we do not have to worry about the Down face corner cubies yet). Also observe that all of the edge cubies are also left in the same position and orientation except for 3 of the Down face edge cubies, y_{10}, y_{11} and y_{12}. If the cube is in the first configuration of no edge cubies in the right orientation then this move must be performed 3 times to get to the final configuration of all four cubies oriented, being held as the figures in the diagram show with each R[DF]R^{-1} move. During the first move it does not matter how your Rubik's Cube is held, as long as the Bottom layer is the Down face. Once the Cube is in the second configuration R[DF]R^{-1} needs to be performed twice more, hold the Rubik's Cube during the first iteration of R[DF]R^{-1} such that the edge cubies that match the middle color are in the **df** and **dl** cubicles. Then you will either start, or have performed the moves such that the Cube is in the third configureation and the move R[DF]R^{-1} needs to be preformed once more. When you perform the move in this position be sure to hold the cube such that the edge cubies that match the color of the middle Down face cubie are in the **dl** and **db** cubicles. By doing this move the desired amount of times it will result in the fourth configuration, where all of the Down face edge cubies match the middle cubie color. Note that it is very important to hold the Rubik's Cube such that the Down face cubies are in the right position in respect to the Front and Right faces, otherwise the fourth configuration will not result.



Now that the Down face edge cubies have the right orientation, we have to get them into the right position so they are lined up with the correct colors on the Front, Left, Back, and Right faces. First rotate the Down face until at least one face has an edge cubie lined up with the same color on one of the side faces (it will already have the correct color on the down face since we just oriented them in the last step). Note from an earlier exercise that D does not affect the orientations of any of the edge cubies, only the positions of the Down face edge and corner cubies. Once you have an edge cubie lined up with its side color call this face F, such that the lined up edge cubie is in the **df** cubicle. Note, if you already have two adjacent faces where the edge cubies lined up you can skip this step. With your positioned cubie on the front face perform the move $F^{-1}D^{-1}FD^{-1}F^{-1}(D^{-1})^2FD^{-1}D^{-1}$. Shown in Table 8, this move does not affect any of the upper or middle corner and edge cubies in their orientation or position. Also the orientation of all of the edge cubies stays the same, which is exactly what we want since we have already oriented them and just need to change their positions.

From that move, we should now have two adjacent sides in the right position as well as the right orientation. When this is the case rotate your entire Rubik's Cube so one aligned edge is on the Back face, in the **db** cubicle, and the other is on the Left face, in the **dl** cubicle. Then, with the edge cubies that are in the wrong position on the Right and Left faces, perform the move $\mathrm{D}^{-1}\mathrm{R}^{-1}\mathrm{D}^{-1}\mathrm{R}\mathrm{D}^{-1}\mathrm{R}^{-1}(\mathrm{D}^{-1})^2\mathrm{R}$. In Table 9, we see that the only edge cubies whose positions change are in the positions 10 and 11, which is **dr** and **df**, the two edge cubies out of position. Now we have all of the edge cubies in the right position and orientation.

Next we can begin to look at the corner cubies on the Down face. If a corner cubie is already in the correct position and oriented this next step is not necessary. However, if all four corners are in the wrong positions and orientations an extra step must be taken. The move $\mathrm{R}^{-1}\mathrm{D}\mathrm{L}\mathrm{D}^{-1}\mathrm{R}\mathrm{D}\mathrm{L}^{-1}\mathrm{D}^{-1}$ must be performed if all four corners cubies are in the wrong positions. As seen by Table 10, the position and orientation of all of the edge cubies stay the same but the 4 Down face corner cubies all change position and two change orientation. If for some reason doing this move once does not result in a Down face corner cubie being in the correct orientation and position then we can repeat this move until one does since, as seen in Table 10, the corner cubies continue to move into different positions and orientations.

From this point forward, in order to minimize potential mistakes, flip the Rubik's Cube so the bottom layer, that still has unoriented cubies, is now the Up face, and the other two solved layers are the bottom two layers. Thus, now we will have all unsolved cubies on the Up face and we will be talking about the Up face cubicles that we are moving the cubies into. Once we have at least one corner in the correct orientation and position we need to look and see if the three remaining corner cubies are in the correct positions. If they are already in the correct position (don't worry about orientations just yet) then this step can be skipped and we only have to worry about finding the correct orientation. In order to get the other cubies into their correct cubicles we will perform the move $\mathrm{U}\mathrm{R}\mathrm{U}^{-1}\mathrm{L}^{-1}\mathrm{U}\mathrm{R}^{-1}\mathrm{U}^{-1}\mathrm{L}$ with the already positioned cubie starting in the **urf** or 2 corner cubie position. In Table 11 notice how 2 and x_2, or **urf** stay in the same position and orientation, along with all of the Down face corner cubies and all of the edge cubies in every layer. If all of the cubies are still not in the correct position, perform the move again.

After getting all of the corner cubies in the right position and at least one corner into the correct position on that layer, there are three different possible configurations that the Rubik's Cube could be in. It could have two adjacent unoriented corner cubies, three unoriented corner cubies, or two unoriented corners opposite of each other. The methods of reorienting the three different orientations of these final cubies are very similar and explained below. It is important to note that in all 3 configuration solutions the commutator of $[\mathrm{R}^{-1}, \mathrm{D}^{-1}]$ is used 6 or 12 times. Recall from an earlier exercise that the commutator has an order of 6. This means that we are using the commutator back to its original configuration where all of the rest of the cubies we have already solved are all in the same spot. The specific moves in between the commutator moves are utilized to reorient the needed unoriented corner cubies. Note, we are still working with the unoriented corner cubies on the Up face of the Rubik's Cube. Let us look at the three different possible configuration solutions,

1. **Two Adjacent Corner Cubies**

   In this configuration have the two unoriented cubies on the Up face, as stated above, and in the cubicles **urf** and **ufl** for both of these cases.

   (a) **Case 1** The desired color needed on the Up face is in the 1 orientation face for **urf** and **ufl**

   If this is the case apply the move $R^{-1}D^{-1}RDR^{-1}D^{-1}RDU^{-1}R^{-1}D^{-1}RD\,R^{-1}D^{-1}RDR^{-1}D^{-1}RDR^{-1}D^{-1}RDU$ which equals $[R^{-1}, D^{-1}]^2U^{-1}[R^{-1}, D^{-1}]^4U$. This is a total of 6 commutators, which brings us back to the starting orientation, paired with U and U^{-1}, which also brings us back to the starting orientation. Since the whole move is combined of moves that only result in the starting configuration this allows the move to keep the positions of the cubes the same while only changing the orientations. This move changes the orientation of both of the cubies by two twists. By twisting the orientation twice we move the colored faces from the 1 orientated face to the 0 orientated face, or the Up face, giving us the whole correct configuration of the Rubik's Cube. This is proved in Table 12, where we see that the only two aspects that are affected are the orientations of x_1 and x_2, **urf** and **ufl**.

   (b) **Case 2** The desired color needed on the Up face is in the 2 orientation face for **urf** and **ufl**

   If this is the case apply the move $R^{-1}D^{-1}RDR^{-1}D^{-1}RDR^{-1}D^{-1}RD\,R^{-1}D^{-1}RDU^{-1}R^{-1}D^{-1}RDR^{-1}D^{-1}RDU$ which equals $[R^{-1}, D^{-1}]^4U^{-1}[R^{-1}, D^{-1}]^2U$. This move changes the orientation of both of the cubies by only one twist. Because we switched where the U^{-1} move is, this changed the end orientation by making the cubies twist one less time in their oriented positions. By twisting the orientation only once this time we move the colored faces from the 2 orientated face to the 0 orientated face, or the Up face, resulting in the solved Rubik's Cube. This is proved in Table 13, where we see that the only two aspects that are affected are the orientations of x_1 and x_2, **urf** and **ufl**.

2. **Three Corner Cubies**

   In this configuration have the three unoriented cubies on the Up face, as stated above, and in the cubicles **ubr**, **urf**, and **ufl** for both cases.

(a) **Case 1** The desired color needed on the Up face is in the 1 orientation face for **ubr**, **urf**, and **ufl**

When we begin this step with this orientation we want to apply the move $\mathrm{R^{-1}D^{-1}RDR^{-1}D^{-1}RDUR^{-1}D^{-1}RDR^{-1}D^{-1}RDUUR^{-1}D^{-1}RDR^{-1}D^{-1}}$ RDU, which equals $[\mathrm{R^{-1}, D^{-1}}]^2\mathrm{U}[\mathrm{R^{-1}, D^{-1}}]^2\mathrm{U}^2[\mathrm{R^{-1}, D^{-1}}]^2\mathrm{U}$. Again we have have groups of moves that bring the cubies back to the same positions they started in, the 6 $[\mathrm{R^{-1}, D^{-1}}]$ commutators and the 4 U moves. In Table 14 we see that x_1, x_2, and x_3, or **ubr**, **urf**, and **ufl**, are twisted twice in their orientations, resulting in a solved configuration of the Rubik's Cube.

(b) **Case 2** The desired color needed on the Up face is in the 2 orientation face for **ubr**, **urf**, and **ufl**

If the Rubik's Cube is in this configuration we must perform the longest move mentioned in this project. For this configuration we must perform the move $\mathrm{R^{-1}D^{-1}RDR^{-1}D^{-1}RDR^{-1}D^{-1}RDR^{-1}D^{-1}RDUR^{-1}D^{-1}R\,DR^{-1}D^{-1}RDR^{-1}}$ $\mathrm{R^{-1}D^{-1}RDU}$, which equals $[\mathrm{R^{-1}, D^{-1}}]^4\mathrm{U}[\mathrm{R^{-1}, D^{-1}}]^4\mathrm{U}^2[\mathrm{R^{-1}, D^{-1}}]^4\mathrm{U}$. This move is also different from the rest of these end moves because it has 12 $[\mathrm{R^{-1}, D^{-1}}]$ commutators instead of 6. This still brings the cubies back to their starting configuration because 12 is a multiple of 6. In Table 15 and 16 (the moves had to be separated in order to be more easily read) we see that x_1, x_2, and x_3, or **ubr**, **urf**, and **ufl**, are twisted once in their orientations, resulting in a completed Rubik's Cube.

3. **Two Opposite Corner Cubies**

In this configuration have the two unoriented cubies on the Up face, as stated above, and in the cubicles **urb** and **ufl**. Given this configuration we can apply the move $\mathrm{UR^{-1}D^{-1}RDR^{-1}D^{-1}RDUUR^{-1}D^{-1}RDR^{-1}D^{-1}RDR^{-1}D^{-1}RD}$ $\mathrm{R^{-1}D^{-1}RDU}$, which equals $\mathrm{U}[\mathrm{R^{-1}, D^{-1}}]^2\mathrm{U}^2[\mathrm{R^{-1}, D^{-1}}]^4\mathrm{U}$. Note again that we have groups of moves that bring the cubies back to the same positions they started in, the 6 $[\mathrm{R^{-1}, D^{-1}}]$ commutators and the 4 U moves. Thus in this move we are only affecting the orientation, not the position. As we can see in Table 17, this move only changes the orientation of x_1 and x_3, or **urb** and **ufl**. Thus, using this move we can change just those two cubies orientations by one twist, giving us the desired solved configuration of the Rubik's Cube.

5 Bibliography

References

[1] Janet Chen. *Group Theory and the Rubik's Cube.*

[2] Raymond Tran *A Mathematical Approach To Solving Rubik's Cube.* UBC Math308 – Fall 200.

6 Appendix

Table 2: $FU^{-1}RU$

σ	F	U^{-1}	R	U		x	F	U^{-1}	R	U
1	6	4	4	2		0	1	0	0	2
2	1	6	2	6		0	2	1	2	3
3	3	1	6	3		0	0	2	3	0
4	4	3	3	4		0	0	0	0	0
5	5	5	5	5		0	0	0	0	0
6	7	7	7	7		0	2	2	2	2
7	2	2	8	8		0	1	1	2	2
8	8	8	1	1		0	0	0	3	3
τ						y				
1	1	2	2	1		0	0	0	0	0
2	2	8	3	2		0	0	1	1	0
3	8	4	4	3		0	1	0	0	1
4	4	1	1	4		0	0	0	0	0
5	5	5	5	5		0	0	0	0	0
6	6	6	8	8		0	0	0	1	1
7	3	3	10	10		0	1	1	0	0
8	11	11	11	11		0	1	1	1	1
9	9	9	9	9		0	0	0	0	0
10	10	10	6	6		0	0	0	0	0
11	7	7	7	7		0	1	1	1	1
12	12	12	12	12		0	0	0	0	0

Table 3: $F^{-1}R^{-1}D^{-1}RFF$

σ	F^{-1}	R^{-1}	D^{-1}	R	F	F		x	F^{-1}	R^{-1}	D^{-1}	R	F	F
1	2	2	2	2	7	5		0	-2	-2	-2	-2	-1	5
2	7	3	3	6	2	7		0	-1	-2	-2	-3	0	1
3	3	8	8	3	3	3		0	0	-1	-1	0	0	0
4	4	4	4	4	4	4		0	0	0	0	0	0	0
5	5	5	1	1	1	1		0	0	0	-1	-1	-1	-1
6	1	1	7	7	5	6		0	-1	-1	-2	-2	4	0
7	6	7	6	5	6	2		0	-2	-2	-4	2	-2	1
8	8	6	5	8	8	8		0	0	-4	0	0	0	0
τ								y						
1	1	1	1	1	1	1		0	0	0	0	0	0	0
2	2	6	6	2	2	2		0	0	0	0	0	0	0
3	7	7	7	7	3	11		0	-1	-1	-1	-1	0	1
4	4	4	4	4	4	4		0	0	0	0	0	0	0
5	5	5	5	5	5	5		0	0	0	0	0	0	0
6	6	10	10	6	6	6		0	0	0	0	0	0	0
7	11	2	2	9	7	3		0	-1	0	0	0	0	1
8	3	3	3	3	11	9		0	-1	-1	-1	-1	0	2
9	9	9	12	12	12	12		0	0	0	0	0	0	0
10	10	11	9	10	10	10		0	0	-1	0	0	0	0
11	8	8	11	11	9	7		0	-1	-1	-1	-1	1	1
12	12	12	8	8	8	8		0	0	0	-1	-1	-1	-1

Table 4: $R^{-1}D^{-1}RFF$

σ	R^{-1}	D^{-1}	R	F	F		x	R^{-1}	D^{-1}	R	F	F
1	1	1	1	2	5		0	0	0	0	0	5
2	3	3	7	1	2		0	-2	-2	-1	2	2
3	8	8	3	3	3		0	-1	-1	0	0	0
4	4	4	4	4	4		0	0	0	0	0	0
5	5	6	6	6	6		0	0	0	0	0	0
6	6	2	2	5	7		0	0	-1	-1	4	2
7	2	7	5	7	1		0	-1	-2	2	0	3
8	7	5	8	8	8		0	-2	0	0	0	0
τ							y					
1	1	1	1	1	1		0	0	0	0	0	0
2	6	6	2	2	2		0	0	0	0	0	0
3	3	3	3	8	7		0	0	0	0	1	0
4	4	4	4	4	4		0	0	0	0	0	0
5	5	5	5	5	5		0	0	0	0	0	0
6	10	10	6	6	6		0	0	0	0	0	0
7	2	2	9	3	8		0	0	0	0	1	0
8	8	8	8	7	9		0	0	0	0	1	0
9	9	12	12	12	12		0	0	0	0	0	0
10	7	9	10	10	10		0	0	0	0	0	0
11	11	7	7	9	3		0	0	0	0	1	0
12	12	11	11	11	11		0	0	0	0	0	0

Table 5: $D^{-1}R^{-1}DRDFD^{-1}F^{-1}$

σ	D^{-1}	R^{-1}	D	R	D	F	D^{-1}	F^{-1}		x	D^{-1}	R^{-1}	D	R	D	F	D^{-1}	F^{-1}
1	1	1	1	1	1	8	8	1		0	0	0	0	0	0	-1	-1	0
2	2	3	3	7	7	1	1	2		0	0	-2	-2	1	1	2	2	0
3	3	5	5	3	3	3	3	3		0	0	-1	-1	0	0	0	0	0
4	4	4	4	4	4	4	4	4		0	0	0	0	0	0	0	0	0
5	6	6	8	8	5	5	6	6		0	0	0	-2	-2	0	0	2	2
6	7	7	6	6	8	6	7	8		0	0	0	0	0	-2	2	2	-2
7	8	2	7	2	6	7	2	7		0	0	-1	0	1	0	2	1	0
8	5	8	2	5	2	2	5	5		0	0	-2	-1	0	1	1	0	0
τ										y								
1	1	1	1	1	1	1	1	1		0	0	0	0	0	0	0	0	0
2	2	6	6	2	2	2	2	2		0	0	0	0	0	0	0	0	0
3	3	3	3	3	3	8	8	3		0	0	0	0	0	0	1	1	0
4	4	4	4	4	4	4	4	4		0	0	0	0	0	0	0	0	0
5	5	5	5	5	5	5	5	5		0	0	0	0	0	0	0	0	0
6	6	9	9	6	6	6	6	6		0	0	0	0	0	0	0	0	0
7	7	2	2	10	10	3	3	11		0	0	0	0	0	0	1	1	-1
8	8	8	8	8	8	12	12	8		0	0	0	0	0	0	1	1	0
9	12	12	7	7	9	9	7	7		0	0	0	0	0	0	0	0	0
10	9	7	10	9	11	11	9	9		0	0	0	0	0	0	0	0	0
11	10	10	11	11	12	10	11	12		0	0	0	0	0	0	1	0	0
12	11	11	12	12	7	7	10	10		0	0	0	0	0	0	0	1	1

Table 6: $DFD^{-1}F^{-1}D^{-1}R^{-1}DR$

σ	D	F	D^{-1}	F^{-1}	D^{-1}	R^{-1}	D	R		x	D	F	D^{-1}	F^{-1}	D^{-1}	R^{-1}	D	R
1	1	5	5	1	1	1	1	1		0	0	1	1	0	0	0	0	0
2	2	1	1	7	7	3	3	2		0	0	2	2	-1	-1	-2	-2	0
3	3	3	3	3	3	6	6	3		0	0	0	0	0	0	1	1	0
4	4	4	4	4	4	4	4	4		0	0	0	0	0	0	0	0	0
5	8	8	6	6	5	5	8	8		0	0	0	2	2	0	0	-2	-2
6	5	6	2	5	2	2	5	5		0	0	2	1	0	-1	-1	0	0
7	6	2	7	2	8	7	2	7		0	0	1	0	-1	0	-2	-1	0
8	7	7	8	8	6	8	7	6		0	0	0	0	0	2	-2	-2	2
τ										y								
1	1	1	1	1	1	1	1	1		0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	6	6	2		0	0	0	0	0	0	0	0	0
3	3	8	8	3	3	3	3	3		0	0	1	1	0	0	0	0	0
4	4	4	4	4	4	4	4	4		0	0	0	0	0	0	0	0	0
5	5	5	5	5	5	5	5	5		0	0	0	0	0	0	0	0	0
6	6	6	6	6	6	9	9	6		0	0	0	0	0	0	0	0	0
7	7	3	3	11	11	2	2	10		0	0	1	1	-1	-1	0	0	0
8	8	12	12	8	8	8	8	8		0	0	1	1	0	0	0	0	0
9	10	10	9	9	7	7	11	11		0	0	0	0	0	1	1	-1	-1
10	11	11	10	10	9	11	10	9		0	0	0	0	0	0	-1	0	0
11	12	7	11	12	10	10	12	12		0	0	1	0	0	0	0	0	0
12	9	9	7	7	12	12	7	7		0	0	0	1	1	0	0	1	1

Table 7: $RDFD^{-1}F^{-1}R^{-1}$

σ	R	D	F	D^{-1}	F^{-1}	R^{-1}		x	R	D	F	D^{-1}	F^{-1}	R^{-1}
1	1	1	5	5	1	1		0	0	0	1	1	0	0
2	7	7	1	1	8	2		0	1	1	2	2	1	0
3	2	2	2	2	2	3		0	2	2	2	2	2	0
4	4	4	4	4	4	4		0	0	0	0	0	0	0
5	5	3	3	6	6	6		0	0	1	1	2	2	2
6	6	5	6	7	5	5		0	0	0	2	2	0	0
7	8	6	7	8	7	8		0	2	0	2	2	0	0
8	3	8	8	3	3	7		0	1	2	2	1	1	-2
τ								y						
1	1	1	1	1	1	1		0	0	0	0	0	0	0
2	7	7	7	7	7	2		0	0	0	0	0	0	0
3	3	3	8	8	3	3		0	0	0	1	1	0	0
4	4	4	4	4	4	4		0	0	0	0	0	0	0
5	5	5	5	5	5	5		0	0	0	0	0	0	0
6	2	2	2	2	2	6		0	0	0	0	0	0	0
7	10	10	3	3	11	7		0	0	0	1	1	-1	0
8	8	8	12	12	8	8		0	0	0	1	1	0	0
9	9	6	6	9	9	9		0	0	0	0	0	0	0
10	6	11	11	6	6	11		0	0	0	0	0	0	-1
11	11	12	10	11	12	12		0	0	0	1	0	0	0
12	12	9	9	10	10	10		0	0	0	0	1	1	1

Table 8: $F^{-1}D^{-1}FD^{-1}F^{-1}(D^{-1})^2FD^{-1}D^{-1}$

σ	F^{-1}	D^{-1}	F	D^{-1}	F^{-1}	D^{-1}	D^{-1}	F	D^{-1}		x	F^{-1}	D^{-1}	F	D^{-1}	F^{-1}	D^{-1}	D^{-1}	F	D^{-1}
1	2	2	6	6	2	2	2	1	1		0	-2	-2	-1	-1	-2	-2	-2	0	0
2	7	7	2	2	5	5	5	2	2		0	-1	-1	0	0	-1	-1	-1	0	0
3	3	3	3	3	3	3	3	3	3		0	0	0	0	0	0	0	0	0	0
4	4	4	4	4	4	4	4	4	4		0	0	0	0	0	0	0	0	0	0
5	5	1	1	8	8	6	7	7	8		0	0	-1	-1	2	2	-2	-2	-2	4
6	1	6	8	7	6	7	1	8	5		0	-1	-2	2	0	-2	-2	-1	4	0
7	6	8	7	5	7	1	8	5	6		0	-2	0	0	0	-2	-1	2	0	-2
8	8	5	5	1	1	8	6	6	7		0	0	0	0	-1	-1	2	-2	-2	-2
τ											y									
1	1	1	1	1	1	1	1	1	1		0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2		0	0	0	0	0	0	0	0	0	0
3	7	7	3	3	7	7	7	3	3		0	0	0	0	0	0	0	0	0	0
4	4	4	4	4	4	4	4	4	4		0	0	0	0	0	0	0	0	0	0
5	5	5	5	5	5	5	5	5	5		0	0	0	0	0	0	0	0	0	0
6	6	6	6	6	6	6	6	6	6		0	0	0	0	0	0	0	0	0	0
7	11	11	7	7	9	9	9	7	7		0	-1	-1	0	0	-1	-1	-1	0	0
8	3	3	10	10	3	3	3	8	8		0	-1	-1	0	0	-1	-1	-1	0	0
9	9	12	12	8	8	11	10	10	12		0	0	0	0	0	0	0	0	0	0
10	10	9	9	12	12	8	11	11	10		0	0	0	0	0	0	0	0	0	0
11	8	10	11	9	10	12	8	9	11		0	0	0	0	0	0	0	0	0	0
12	12	8	8	11	11	10	12	12	9		0	0	0	0	0	0	0	0	0	0

Table 9: $\mathrm{D}^{-1}\mathrm{R}^{-1}\mathrm{D}^{-1}\mathrm{R}\mathrm{D}^{-1}\mathrm{R}^{-1}(\mathrm{D}^{-1})^2\mathrm{R}$

σ	D^-1	R^-1	D^-1	R	D^-1	R^-1	D^-1	D^-1	R		x	D^-1	R^-1	D^-1	R	D^-1	R^-1	D^-1	D^-1	R
1	1	1	1	1	1	1	1	1	1		0	0	0	0	0	0	0	0	0	0
2	2	3	3	8	8	3	3	3	2		0	0	-2	-2	-1	-1	-2	-2	-2	0
3	3	5	5	3	3	7	7	7	3		0	0	-1	-1	0	0	-1	-1	-1	0
4	4	4	4	4	4	4	4	4	4		0	0	0	0	0	0	0	0	0	0
5	6	6	7	7	2	2	6	8	8		0	0	0	0	0	-1	-1	2	-2	-2
6	7	7	2	2	6	6	8	5	5		0	0	0	-1	-1	2	2	-2	-2	-2
7	8	2	8	6	5	8	5	2	6		0	0	-1	-2	2	0	-2	-2	-1	4
8	5	8	6	5	7	5	2	6	7		0	0	-2	0	0	0	-2	-1	2	0
τ											y									
1	1	1	1	1	1	1	1	1	1		0	0	0	0	0	0	0	0	0	0
2	2	6	6	2	2	6	6	6	2		0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	3	3	3	3	3		0	0	0	0	0	0	0	0	0	0
4	4	4	4	4	4	4	4	4	4		0	0	0	0	0	0	0	0	0	0
5	5	5	5	5	5	5	5	5	5		0	0	0	0	0	0	0	0	0	0
6	6	9	9	6	6	11	11	11	6		0	0	0	0	0	0	0	0	0	0
7	7	2	2	12	12	2	2	2	7		0	0	0	0	0	0	0	0	0	0
8	8	8	8	8	8	8	8	8	8		0	0	0	0	0	0	0	0	0	0
9	12	12	11	11	10	10	7	9	9		0	0	0	0	0	0	0	0	0	0
10	9	7	12	9	11	12	10	7	11		0	0	0	0	0	0	0	0	0	0
11	10	10	7	7	9	9	12	10	10		0	0	0	0	0	0	0	0	0	0
12	11	11	10	10	7	7	9	12	12		0	0	0	0	0	0	0	0	0	0

Table 10: $R^{-1}DLD^{-1}RDL^{-1}D^{-1}$

σ	R^{-1}	D	L	D^{-1}	R	D	L^{-1}	D^{-1}		x	R^{-1}	D	L	D^{-1}	R	D	L^{-1}	D^{-1}
1	1	1	4	4	4	4	1	1		0	0	0	2	2	2	2	0	0
2	3	3	3	3	2	2	2	2		0	-2	-2	-2	-2	0	0	0	0
3	8	8	8	8	3	3	3	3		0	-1	-1	-1	-1	0	0	0	0
4	4	4	7	7	7	7	4	4		0	0	0	-1	-1	-1	-1	0	0
5	5	7	5	1	1	8	7	8		0	0	-2	2	1	1	0	-2	-2
6	6	5	1	6	6	1	8	6		0	0	0	1	0	0	1	-2	0
7	2	6	6	2	5	6	6	5		0	-1	0	0	-1	4	0	0	4
8	7	2	2	5	8	5	5	7		0	-2	-1	-1	2	0	4	4	-2
τ										y								
1	1	1	1	1	1	1	1	1		0	0	0	0	0	0	0	0	0
2	6	6	6	6	2	2	2	2		0	0	0	0	0	0	0	0	0
3	3	3	3	3	3	3	3	3		0	0	0	0	0	0	0	0	0
4	4	4	5	5	5	5	4	4		0	0	0	0	0	0	0	0	0
5	5	5	9	9	9	9	5	5		0	0	0	0	0	0	0	0	0
6	10	10	10	10	6	6	6	6		0	0	0	0	0	0	0	0	0
7	2	2	2	2	7	7	7	7		0	0	0	0	0	0	0	0	0
8	8	8	4	4	4	4	8	8		0	0	0	0	0	0	0	0	0
9	9	7	7	8	8	10	10	9		0	0	0	0	0	0	0	0	0
10	7	11	11	7	10	11	11	10		0	0	0	0	0	0	0	0	0
11	11	12	12	11	11	12	12	11		0	0	0	0	0	0	0	0	0
12	12	9	8	12	12	8	9	12		0	0	0	0	0	0	0	0	0

Table 11: $URU^{-1}L^{-1}UR^{-1}U^{-1}L$

σ	U	R	U^{-1}	L^{-1}	U	R^{-1}	U^{-1}	L		x	U	R	U^{-1}	L^{-1}	U	R^{-1}	U^{-1}	L
1	2	2	1	6	2	2	6	4		0	0	0	0	-1	0	0	-1	2
2	3	7	2	2	7	1	2	2		0	0	1	0	0	1	-4	0	0
3	4	3	7	7	1	4	1	1		0	0	2	1	1	-2	0	-4	-4
4	1	1	3	1	6	6	4	3		0	0	0	2	-2	-1	-1	0	2
5	5	5	5	3	3	3	3	5		0	0	0	0	1	1	1	1	0
6	6	6	6	5	5	5	5	6		0	0	0	0	-2	-2	-2	-2	0
7	7	8	8	8	8	7	7	7		0	0	2	2	2	2	0	0	0
8	8	4	4	4	4	8	8	8		0	0	1	1	1	1	0	0	0
τ										y								
1	4	4	7	7	8	8	1	1		0	0	0	0	0	0	0	0	0
2	1	7	2	2	7	1	2	2		0	0	0	0	0	0	0	0	0
3	2	2	3	3	2	2	3	3		0	0	0	0	0	0	0	0	0
4	3	3	4	8	3	3	8	4		0	0	0	0	0	0	0	0	0
5	5	5	5	4	4	4	4	5		0	0	0	0	0	0	0	0	0
6	6	1	1	1	1	6	6	6		0	0	0	0	0	0	0	0	0
7	7	10	10	10	10	7	7	7		0	0	0	0	0	0	0	0	0
8	8	8	8	12	12	12	12	8		0	0	0	0	0	0	0	0	0
9	9	9	9	9	9	9	9	9		0	0	0	0	0	0	0	0	0
10	10	6	6	6	6	10	10	10		0	0	0	0	0	0	0	0	0
11	11	11	11	11	11	11	11	11		0	0	0	0	0	0	0	0	0
12	12	12	12	5	5	5	5	12		0	0	0	0	0	0	0	0	0

Table 12: $R^{-1}D^{-1}RDR^{-1}D^{-1}RDU^{-1}R^{-1}D^{-1}RDR^{-1}D^{-1}RDR^{-1}D^{-1}RDR^{-1}D^{-1}RDU$

σ	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U^{-1}	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U
1	1	1	1	1	1	1	1	1	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	1
2	3	3	7	7	3	3	2	2	1	2	2	7	7	2	2	1	1	2	2	7	7	2	2	1	1	2
3	8	8	3	3	5	5	3	3	2	8	8	2	2	5	5	2	2	8	8	2	2	5	5	2	2	3
4	4	4	4	4	4	4	4	4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4
5	5	6	6	8	8	6	6	5	5	5	6	6	8	8	6	6	5	5	6	6	8	8	6	6	5	5
6	6	2	2	6	6	7	7	6	6	6	1	1	6	6	7	7	6	6	1	1	6	6	7	7	6	6
7	2	7	5	2	7	2	8	7	7	1	7	5	1	7	1	8	7	1	7	5	1	7	1	8	7	7
8	7	5	8	5	2	8	5	8	8	7	5	8	5	1	8	5	8	7	5	8	5	1	8	5	8	8
x																										
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-4
0	-2	-2	-1	-1	-2	-2	-2	-2	0	-4	-4	-3	-3	-4	-4	-2	-2	-4	-4	-5	-5	-4	-4	-4	-4	-2
0	-1	-1	0	0	1	1	0	0	-2	1	1	-2	-2	3	3	-2	-2	3	3	-2	-2	5	5	-2	-2	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2	2	2	0	0	2	2	0	0	4	4	0	0	4	4	0	0	6	6
0	0	-1	-1	0	0	-2	-2	0	0	0	-1	-1	0	0	-4	-4	0	0	-3	-3	0	0	-6	-6	0	0
0	-1	-2	2	-1	-2	-3	2	-2	-2	-1	-4	4	-1	-4	-3	4	-4	-3	-6	6	-3	-6	-5	6	-6	-6
0	-2	0	0	2	-3	0	2	2	2	-4	2	2	4	-3	2	4	4	-6	4	4	6	-5	4	6	6	6
τ																										
1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1
2	6	6	2	2	6	6	2	2	3	6	6	3	3	6	6	3	3	6	6	3	3	6	6	3	3	2
3	3	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3
4	4	4	4	4	4	4	4	4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	10	10	6	6	7	7	6	6	6	9	9	6	6	10	10	6	6	7	7	6	6	9	9	6	6	6
7	2	2	9	9	2	2	10	10	10	3	3	7	7	3	3	9	9	3	3	10	10	3	3	7	7	7
8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
9	9	12	12	10	10	12	12	7	7	7	12	12	9	9	12	12	10	10	12	12	7	7	12	12	9	9
10	7	9	10	7	9	10	7	9	9	10	7	9	10	7	9	10	7	9	10	7	9	10	7	9	10	10
11	11	7	7	11	11	9	9	11	11	11	10	10	11	11	7	7	11	11	9	9	11	11	10	10	11	11
12	12	11	11	12	12	11	11	12	12	12	11	11	12	12	11	11	12	12	11	11	12	12	11	11	12	12
y																										
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 13: $R^{-1}D^{-1}RDR^{-1}D^{-1}RDR^{-1}D^{-1}RDR^{-1}D^{-1}RDU^{-1}R^{-1}D^{-1}RDR^{-1}D^{-1}RDU$

σ	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U^{-1}	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	4	4	4	4	4	4	4	4	4	1
2	3	3	7	7	3	3	2	2	3	3	7	7	3	3	2	2	1	2	2	7	7	2	2	1	1	2
3	8	8	3	3	5	5	3	3	8	8	3	3	5	5	3	3	2	8	8	2	2	5	5	2	2	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3	3	3	3	3	3	3	3	3	4
5	5	6	6	8	8	6	6	5	5	6	6	8	8	6	6	5	5	5	6	6	8	8	6	6	5	5
6	6	2	2	6	6	7	7	6	6	2	2	6	6	7	7	6	6	6	1	1	6	6	7	7	6	6
7	2	7	5	2	7	2	8	7	2	7	5	2	7	2	8	7	7	1	7	5	1	7	1	8	7	7
8	7	5	8	5	2	8	5	8	7	5	8	5	2	8	5	8	8	7	5	8	5	1	8	5	8	8
x																										
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2
0	-2	-2	-1	-1	-2	-2	-2	-2	-2	-2	-3	-3	-2	-2	-4	-4	0	-6	-6	-5	-5	-6	-6	-2	-2	-4
0	-1	-1	0	0	1	1	0	0	1	1	0	0	3	3	0	0	-4	3	3	-4	-4	5	5	-4	-4	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2	2	0	0	2	2	0	0	4	4	4	0	0	4	4	0	0	6	6
0	0	-1	-1	0	0	-2	-2	0	0	-3	-3	0	0	-4	-4	0	0	0	-1	-1	0	0	-6	-6	0	0
0	-1	-2	2	-1	-2	-3	2	-2	-3	-4	4	-3	-4	-5	4	-4	-4	-1	-6	6	-1	-6	-3	6	-6	-6
0	-2	0	0	2	-3	0	2	2	-4	2	2	4	-5	2	4	4	4	-6	4	4	6	-3	4	6	6	6
τ																										
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	1
2	6	6	2	2	6	6	2	2	6	6	2	2	6	6	2	2	3	6	6	3	3	6	6	3	3	2
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4	4	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	1	1	1	1	1	1	1	1	1	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	10	10	6	6	7	7	6	6	9	9	6	6	10	10	6	6	6	7	7	6	6	9	9	6	6	6
7	2	2	9	9	2	2	10	10	2	2	7	7	2	2	9	9	9	3	3	10	10	3	3	7	7	7
8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
9	9	12	12	10	10	12	12	7	7	12	12	9	9	12	12	10	10	10	12	12	7	7	12	12	9	9
10	7	9	10	7	9	10	7	9	10	7	9	10	7	9	10	7	7	9	10	7	9	10	7	9	10	10
11	11	7	7	11	11	9	9	11	11	10	10	11	11	7	7	11	11	11	9	9	11	11	10	10	11	11
12	12	11	11	12	12	11	11	12	12	11	11	12	12	11	11	12	12	12	11	11	12	12	11	11	12	12
y																										
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 14: $R^{-1}D^{-1}RDR^{-1}D^{-1}RDUR^{-1}D^{-1}RDR^{-1}D^{-1}RDUUR^{-1}D^{-1}RDR^{-1}D^{-1}RDU$

σ	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U	U	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U
1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	3	4	4	4	4	4	4	4	4	4	1
2	3	3	7	7	3	3	2	2	3	4	4	7	7	4	4	3	3	4	1	2	2	7	7	2	2	1	1	2
3	8	8	3	3	5	5	3	3	4	8	8	4	4	5	5	4	4	1	2	8	8	2	2	5	5	2	2	3
4	4	4	4	4	4	4	4	4	1	1	1	1	1	1	1	1	1	2	3	3	3	3	3	3	3	3	3	4
5	5	6	6	8	8	6	6	5	5	5	6	6	8	8	6	6	5	5	5	5	6	6	8	8	6	6	5	5
6	6	2	2	6	6	7	7	6	6	6	3	3	6	6	7	7	6	6	6	6	1	1	6	6	7	7	6	6
7	2	7	5	2	7	2	8	7	7	3	7	5	3	7	3	8	7	7	7	1	7	5	1	7	1	8	7	7
8	7	5	8	5	2	8	5	8	8	7	5	8	5	3	8	5	8	8	8	7	5	8	5	1	8	5	8	8
x																												
0	0	0	0	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	0	0	0	0	0	0	0	0	0	-2
0	-2	-2	-1	-1	-2	-2	-2	-2	0	-2	-2	-3	-3	-2	-2	-2	-2	0	0	-4	-4	-5	-5	-4	-4	-2	-2	-2
0	-1	-1	0	0	1	1	0	0	0	1	1	0	0	3	3	0	0	0	-2	3	3	-2	-2	5	5	-2	-2	-2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	0
0	0	0	0	0	0	0	0	2	2	2	0	0	2	2	0	0	4	4	4	4	0	0	4	4	0	0	6	6
0	0	-1	-1	0	0	-2	-2	0	0	0	-1	-1	0	0	-4	-4	0	0	0	0	-1	-1	0	0	-6	-6	0	0
0	-1	-2	2	-1	-2	-3	2	-2	-2	-1	-4	4	-1	-4	-3	4	-4	-4	-4	-1	-6	6	-1	-6	-3	6	-6	-6
0	-2	0	0	2	-3	0	2	2	2	-4	2	2	4	-3	2	4	4	4	4	-6	4	4	6	-3	4	6	6	6
τ																												
1	1	1	1	1	1	1	1	1	4	4	4	4	4	4	4	4	4	3	2	2	2	2	2	2	2	2	2	1
2	6	6	2	2	6	6	2	2	1	6	6	1	1	6	6	1	1	4	3	6	6	3	3	6	6	3	3	2
3	3	3	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2	1	4	4	4	4	4	4	4	4	4	3
4	4	4	4	4	4	4	4	4	3	3	3	3	3	3	3	3	3	2	1	1	1	1	1	1	1	1	1	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	10	10	6	6	7	7	6	6	6	9	9	6	6	10	10	6	6	6	6	7	7	6	6	9	9	6	6	6
7	2	2	9	9	2	2	10	10	10	1	1	7	7	1	1	9	9	9	9	3	3	10	10	3	3	7	7	7
8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
9	9	12	12	10	10	12	12	7	7	7	12	12	9	9	12	12	10	10	10	10	12	12	7	7	12	12	9	9
10	7	9	10	7	9	10	7	9	9	10	7	9	10	7	9	10	7	7	7	9	10	7	9	10	7	9	10	10
11	11	7	7	11	11	9	9	11	11	11	10	10	11	11	7	7	11	11	11	11	9	9	11	11	10	10	11	11
12	12	11	11	12	12	11	11	12	12	12	11	11	12	12	11	11	12	12	12	12	11	11	12	12	11	11	12	12
y																												
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 15: $(R^{-1}D^{-1}RD)^4U(R^{-1}D^{-1}RD)^4UU(R^{-1}D^{-1}RD)^4U$

σ	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	R^{-1}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2
2	3	3	7	7	3	3	2	2	3	3	7	7	3	3	2	2	3	4	4	7	7	4	4	3	3	4
3	8	8	3	3	5	5	3	3	8	8	3	3	5	5	3	3	4	8	8	4	4	5	5	4	4	8
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	1	1	1	1	1	1	1	1	1	1
5	5	6	6	8	8	6	6	5	5	6	6	8	8	6	6	5	5	5	6	6	8	8	6	6	5	5
6	6	2	2	6	6	7	7	6	6	2	2	6	6	7	7	6	6	6	3	3	6	6	7	7	6	6
7	2	7	5	2	7	2	8	7	2	7	5	2	7	2	8	7	7	3	7	5	3	7	3	8	7	3
8	7	5	8	5	2	8	5	8	7	5	8	5	2	8	5	8	8	7	5	8	5	3	8	5	8	7
x																										
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
0	-2	-2	-1	-1	-2	-2	-2	-2	-2	-2	-3	-3	-2	-2	-4	-4	0	-2	-2	-5	-5	-2	-2	-2	-2	-2
0	-1	-1	0	0	1	1	0	0	1	1	0	0	3	3	0	0	0	3	3	0	0	5	5	0	0	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2	2	0	0	2	2	0	0	4	4	4	0	0	4	4	0	0	6	6
0	0	-1	-1	0	0	-2	-2	0	0	-3	-3	0	0	-4	-4	0	0	0	-1	-1	0	0	-6	-6	0	0
0	-1	-2	2	-1	-2	-3	2	-2	-3	-4	4	-3	-4	-5	4	-4	-4	-1	-6	6	-1	-6	-3	6	-6	-3
0	-2	0	0	2	-3	0	2	2	-4	2	2	4	-5	2	4	4	4	-6	4	4	6	-3	4	6	6	-8
τ																										
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	4	4	4	4	4	4	4	4	4	4
2	6	6	2	2	6	6	2	2	6	6	2	2	6	6	2	2	1	6	6	1	1	6	6	1	1	6
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3	3	3	3	3	3	3	3	3	3
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	10	10	6	6	7	7	6	6	9	9	6	6	10	10	6	6	6	7	7	6	6	9	9	6	6	10
7	2	2	9	9	2	2	10	10	2	2	7	7	2	2	9	9	9	1	1	10	10	1	1	7	7	1
8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
9	9	12	12	10	10	12	12	7	7	12	12	9	9	12	12	10	10	10	12	12	7	7	12	12	9	9
10	7	9	10	7	9	10	7	9	10	7	9	10	7	9	10	7	7	9	10	7	9	10	7	9	10	7
11	11	7	7	11	11	9	9	11	11	10	10	11	11	7	7	11	11	11	9	9	11	11	10	10	11	11
12	12	11	11	12	12	11	11	12	12	11	11	12	12	11	11	12	12	12	11	11	12	12	11	11	12	12
y																										
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 16: $(R^{-1}D^{-1}RD)^4U(R^{-1}D^{-1}RD)^4UU(R^{-1}D^{-1}RD)^4U$ Continued

σ	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U	U	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U
	2	2	2	2	2	2	2	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	1
	4	7	7	4	4	3	3	4	1	2	2	7	7	2	2	1	1	2	2	7	7	2	2	1	1	2
	8	4	4	5	5	4	4	1	2	8	8	2	2	5	5	2	2	8	8	2	2	5	5	2	2	3
	1	1	1	1	1	1	1	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4
	6	6	8	8	6	6	5	5	5	5	6	6	8	8	6	6	5	5	6	6	8	8	6	6	5	5
	3	3	6	6	7	7	6	6	6	6	1	1	6	6	7	7	6	6	1	1	6	6	7	7	6	6
	7	5	3	7	3	8	7	7	7	1	7	5	1	7	1	8	7	1	7	5	1	7	1	8	7	7
	5	8	5	3	8	5	8	8	8	7	5	8	5	1	8	5	8	7	5	8	5	1	8	5	8	8
x																										
	-4	-4	-4	-4	-4	-4	-4	-4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-4
	-2	-7	-7	-2	-2	-4	-4	0	0	-6	-6	-9	-9	-6	-6	-2	-2	-6	-6	-11	-11	-6	-6	-4	-4	-4
	5	0	0	7	7	0	0	0	-4	7	7	-4	-4	9	9	-4	-4	9	9	-4	-4	11	11	-4	-4	-4
	0	0	0	0	0	0	0	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	0
	0	0	6	6	0	0	8	8	8	8	0	0	8	8	0	0	10	10	0	0	10	10	0	0	12	12
	-3	-3	0	0	-8	-8	0	0	0	0	-1	-1	0	0	-10	-10	0	0	-3	-3	0	0	-12	-12	0	0
	-8	8	-3	-8	-5	8	-8	-8	-8	-1	-10	10	-1	-10	-3	10	-10	-3	-12	12	-3	-12	-5	12	-12	-12
	6	6	8	-5	6	8	8	8	8	-10	8	8	10	-3	8	10	10	-12	10	10	12	-5	10	12	12	12
τ																										
	4	4	4	4	4	4	4	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1
	6	1	1	6	6	1	1	4	3	6	6	3	3	6	6	3	3	6	6	3	3	6	6	3	3	2
	2	2	2	2	2	2	2	1	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3
	3	3	3	3	3	3	3	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	4
	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	10	6	6	7	7	6	6	6	6	9	9	6	6	10	10	6	6	7	7	6	6	9	9	6	6	6
	1	9	9	1	1	10	10	10	10	3	3	7	7	3	3	9	9	3	3	10	10	3	3	7	7	7
	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
	12	12	10	10	12	12	7	7	7	7	12	12	9	9	12	12	10	10	12	12	7	7	12	12	9	9
	9	10	7	9	10	7	9	9	9	10	7	9	10	7	9	10	7	9	10	7	9	10	7	9	10	10
	7	7	11	11	9	9	11	11	11	11	10	10	11	11	7	7	11	11	9	9	11	11	10	10	11	11
	11	11	12	12	11	11	12	12	12	12	11	11	12	12	11	11	12	12	11	11	12	12	11	11	12	12
y																										
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 17: $UR^{-1}D^{-1}RDR^{-1}D^{-1}RDUUR^{-1}D^{-1}RDR^{-1}D^{-1}RDR^{-1}D^{-1}RDR^{-1}D^{-1}RDU$

σ	U	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U	U	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	R^{-1}	D^{-1}	R	D	U
1	2	2	2	2	2	2	2	2	2	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	1
2	3	4	4	7	7	4	4	3	3	4	1	2	2	7	7	2	2	1	1	2	2	7	7	2	2	1	1	2
3	4	8	8	4	4	5	5	4	4	1	2	8	8	2	2	5	5	2	2	8	8	2	2	5	5	2	2	3
4	1	1	1	1	1	1	1	1	1	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4
5	5	5	6	6	8	8	6	6	5	5	5	5	6	6	8	8	6	6	5	5	6	6	8	8	6	6	5	5
6	6	6	3	3	6	6	7	7	6	6	6	6	1	1	6	6	7	7	6	6	1	1	6	6	7	7	6	6
7	7	3	7	5	3	7	3	8	7	7	7	1	7	5	1	7	1	8	7	1	7	5	1	7	1	8	7	7
8	8	7	5	8	5	3	8	5	8	8	8	7	5	8	5	1	8	5	8	7	5	8	5	1	8	5	8	8
x																												
0	0	0	0	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-4
0	0	-2	-2	-1	-1	-2	-2	-2	-2	0	0	-2	-2	-3	-3	-2	-2	-2	-2	-2	-2	-5	-5	-2	-2	-4	-4	0
0	0	-1	-1	0	0	1	1	0	0	0	0	1	1	0	0	3	3	0	0	3	3	0	0	5	5	0	0	-2
0	0	0	0	0	0	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	0
0	0	0	0	0	0	0	0	0	2	2	2	2	0	0	2	2	0	0	4	4	0	0	4	4	0	0	6	6
0	0	0	-1	-1	0	0	-2	-2	0	0	0	0	-1	-1	0	0	-4	-4	0	0	-3	-3	0	0	-6	-6	0	0
0	0	-1	-2	2	-1	-2	-3	2	-2	-2	-2	-1	-4	4	-1	-4	-3	4	-4	-3	-6	6	-3	-6	-5	6	-6	-6
0	0	-2	0	0	2	-3	0	2	2	2	2	-4	2	2	4	-3	2	4	4	-6	4	4	6	-5	4	6	6	6
τ																												
1	4	4	4	4	4	4	4	4	4	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1
2	1	6	6	1	1	6	6	1	1	4	3	6	6	3	3	6	6	3	3	6	6	3	3	6	6	3	3	2
3	2	2	2	2	2	2	2	2	2	1	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3
4	3	3	3	3	3	3	3	3	3	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	6	10	10	6	6	7	7	6	6	6	6	9	9	6	6	10	10	6	6	7	7	6	6	9	9	6	6	6
7	7	1	1	9	9	1	1	10	10	10	10	3	3	7	7	3	3	9	9	3	3	10	10	3	3	7	7	7
8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
9	9	9	12	12	10	10	12	12	7	7	7	7	12	12	9	9	12	12	10	10	12	12	7	7	12	12	9	9
10	10	7	9	10	7	9	10	7	9	9	9	10	7	9	10	7	9	10	7	9	10	7	9	10	7	9	10	10
11	11	11	7	7	11	11	9	9	11	11	11	11	10	10	11	11	7	7	11	11	9	9	11	11	10	10	11	11
12	12	12	11	11	12	12	11	11	12	12	12	12	11	11	12	12	11	11	12	12	11	11	12	12	11	11	12	12
y																												
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0